Propeller Blade, “Lift”

In the last post I was able to clarify the load torque of the propeller applied through the gearbox to the motor shaft. I haven’t yet covered the main point of the propeller: **LIFT**! A propeller blade is just a spinning wing at each cross-sectional location.

The propeller force described in the NASA diagram in an earlier post summarizes the propeller-induced thrust force but it doesn’t tell us anything about the fluid mechanics around the prop blades.

Wings and Lift

My MIT Fluid Mechanics professor Ain Sonin would not be happy with me if I didn’t remember a very compelling demonstration he performed to illustrate, “streamline curvature” as the lift producer. The link on his name honors his passing in 2010. He died at a young age of 72. I vividly recall his interesting lectures, impeccable sketches, approximation techniques, and his evident love of teaching. Thinking of what I learned from him, and what I should have better absorbed then, I want to cover lift here.

Previously, we considered the torque load as the drag on the motor. It makes sense that if we spin a propeller shaped like the example below the blunt nose and general form of the propeller requires some torque to spin it through the air. This is the drag you feel on your hand when you stick it out of your car window.

Let’s stick with the hand-out-the-window example. Whether we are actual kids or big kids, we all sometimes put our hand out the car window with our palm down. Then we tip our hand just a bit back and it feels like it wants to go up. Is this how a wing or a propeller (a spinning wing) creates lift? No!

A titled hand can rise with the force of air outside the car window but we cannot sustain flight (level-flight) or produce along-axis force from spinning a propeller unless the blades produce, “streamline curvature”! This is something hard to do with your hand out the car window unless you have a very unique shape to your hand and the angles are just right.

You are likely not producing, “lift” with your hand. you are just feeling the drag on an angle and that wants to push your hand up-and-back. If it weren’t attached to your arm it would just flip backwards. If an airplane wing attempted to climb by tipping itself backwards the plane would flip back like a mattress on top of a minivan going down the highway.

It’s about, “Streamline curvature” over the top surface

So what is really going on? In Professor Sonin’s class I was introduced to the fundamentals of Euler’s Equation, and its application in cylindrical coordinates along a, “streamline”. Streamlines are the blue lines flowing around the cross-section of a wing below. Bernoulli’s equation is what we learn to apply in basic fluid mechanics. It can be integrated from Euler’s equation. Euler gives us a deeper level of understanding.

If you look at the cross-sections of the propeller above you can see they look like a bunch of, “wing” cross-sections. The propeller is just a spinning wing. At any moment in time any of those propeller cross-sections above has streamlines flowing over-an-under similar to the simple wing diagram shown here. As it spins, part of the torque supplied by the motor goes into forcing these streamlines to curve around the blade. It is the nature of this curvature that gives us, “lift”!

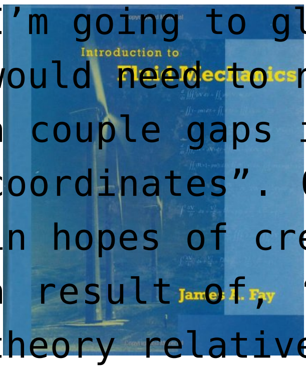
For a wing, the, “weight” in the diagram is the weight of the entire plane, so the wings must produce lift to counter-act that weight, and extra lift to accelerate that weight upwards: to climb. When our propeller spins, we want it to pull a plane forward or a helicopter up (quad-copter in our case). A prop-driven airplane creates lift at the propellers to produce flight speed necessary to produce vertical lift by the wings! The prop, pulls the plane forward with, “lift” from the propellers and the plane stays aloft through, “lift” at the wings. the same principle is applied in two directions!

Our quadcopter will hover when the total lift from the four propellers matches the total weight. It will accelerate upwards or laterally when the total lift exceeds the total weight. The “force” we get out of a wing and a propeller is due to something you can see in the graphic above: the, “streamlines” on the top (B) (the top surface of our quadrotor propellers) have more curvature than the streamlines on the bottom (C).

Euler’s Equation

The text below was in draft when I took Professor Sonin’s course at MIT. I still have a bound copy of the printout. I like this Text and I remember Sonin’s course for providing a more solid mathematical basis for fluid mechanics. Preparing for this post motivated me to (re)learn this material, so I bought a copy of Fay’s book for my shelf.

Recommended Reading



I'm going to gloss-over considerable detail (mostly because I would need to relearn too much right now!). I'm going to fill a couple gaps in the derivations that get us to, "streamline coordinates". Otherwise I'm just organizing references here in hopes of creating a useful guide on this topic of lift as a result of, "streamline curvature". I can't improve the theory relative to these expert sources. I cannot come close to the genius of Euler and the father-son Bernoulli team! I hope I can make it more clear for we mere mortals is all.

Euler's Equations

Euler's Equations for inviscid flow will reveal what causes the lift once we get it into the right form. Inviscid flow is ideal, simplified flow assumed to slip past the wing with no surface interaction (no shear, so no boundary layer, eddies, turbulence)…only the shape of the wing affects the flow, not the surface characteristics.

The Equations will reveal where the $V^2 \Rightarrow \omega^2$ for the propeller speed relationship to output force comes from. I realized I glossed over this topic in the post with the NASA diagram where we saw the simplified equation with the V^2 term. In the last post I resolved some confusion around the motor+gearbox+propeller model for load on the motor. I decided I needed to cover here the entire point of spinning the propeller: "lift" from the blades, or aggregate thrust along the axis of the propeller.

The following PDF extracts key pages from Fay's book (my old draft copy) that lead us to, "Euler's Equation in Streamline Coordinates". I retained a few pages of Bernoulli equation derivation. It is integrated from Euler's equation along a streamline. It is a frequently applied formula. The typically overlooked perpendicular-to-streamline (normal) Euler equation tells us about lift from an airfoil!

James A. Fay: Euler's Equation

Example Application

This tutorial by the late Professor Sonin is excellent. I enjoy his neat sketches and remember them well, as I took notes in his class and attempted to match his detail and neatness.

Sonin: StreamLine Equations of Motion

It is not easy to digest the mathematical symbols, mutlivariable calculus, and terminology unless one has an excellent memory of their studies or practices these topics regularly. I will fill some gaps that troubled me as I re-derived some of the equations.

Cylindrical Coordinates

Although it might look daunting on it’s own, let’s accept Euler’s equation in cartesian (XYZ) coordinates. Then in Fay’s and Sonin’s material above we see we need cylindrical-streamline coordinates to get the equations of motion into some form that will enlighten us on this lift force we so desire.

Step-by-Step Derivation

The key to deriving cylindrical coordinates is to start with a representation of the unit vectors relative to a Cartesian system. I’m going to skip all the diagrams I should place here because you can look it up. I just want to offer a few observations on how elegantly simple the derivations are even thought they look complicated….

We start with radial and angular unit vectors in an X-Y plane, and we know Z in the Cartesian system is same as Z in Cylindrical

$$i_r = cos(\theta(t))i + sin(\theta(t))j$$
$$i_\theta = -sin(\theta(t))i + cos(\theta(t))j$$
$$i_z = i_z$$

A big difference between cylindrical coordinates and Cartesian coordinates is the rotation of the 'r' and ' θ ' axes in time, as our, "fluid particle" moves in θ . Hence showing θ as a function of time in the above equation.

We're going to need derivatives of the unit vectors, and here's what they are:

$\dot{i_r} = (-sin(\theta(t))i + cos(\theta(t))j)\dot{\theta}$ but look at the trig in parenthesis. It's our term for i_θ above so…

$$\dot{i_r} = \dot{\theta} i_\theta$$

And

$\dot{i_\theta} = (-cos(\theta(t))i - sin(\theta(t))j)\dot{\theta}$ but again we see a familiar expression from above so this simplifies to…

$$\dot{i_\theta} = -\dot{\theta} i_r$$

It is these moving axes that give us the, "extra" terms in the cylindrical equations of motion. We get to them simply by taking derivatives of position and velocity.

We start with the position of a particle in cylindrical coordinates.

$P = r \cdot i_r + z \cdot i_z$ By chain rule we differentiate position to velocity as…

$$V=\frac{dP}{dt}=\dot{r}\cdot i_r+r\cdot \dot{i}_r+\dot{z}\cdot i_z+z\cdot \dot{i}_z$$

But we have terms for the derivatives of the unit vectors from above, and $\dot{i}_z=0$ so velocity simplifies to…

$$V=\dot{r}\cdot i_r+r\dot{\theta}i_\theta+\dot{z}\cdot i_z$$

And because I don't want to skip steps we differentiate Velocity to acceleration again by the simple chain rule of differentiation…

$$a=\frac{dV}{dt}=\ddot{r}\cdot i_r+\dot{r}\cdot \dot{i}_r+\dot{r}\dot{\theta}i_\theta+r\ddot{\theta}i_\theta+r\dot{\theta}\dot{i}_\theta+\ddot{z}\cdot i_z+z\cdot \dot{i}_z$$

and we again replace those unit vector derivative terms and $\dot{i}_z=0$ to get, with a bit of rearrangement…

$$a=\frac{dV}{dt}=(\ddot{r}-r\dot{\theta}^2)\cdot i_r+(2\dot{r}\dot{\theta}+r\ddot{\theta})\cdot i_\theta+\ddot{z}\cdot i_z|$$

Remarks on Cylindrical Coordinate Derivation

I must admit that unless I derive these equations as I did here I can fall prey to thinking there is something, "magic" about all those terms in the cylindrical coordinate system compared with our more intuitive Cartesian (XYZ) system. Perhaps for me more than for you I captured the derivation above so I can remind myself there is no magic to it, but simply two rotating unit vectors and the chain-rule of differentiation.

Streamline Coordinates

These are just a slight twist on Cylindrical coordinates. We relate cylindrical coordinates to streamline coordinates as follows…

$i_n = -i_r$ The perpendicular-to-streamline or, “normal” to flow unit vector.

$i_s = i_\theta$ The along-path unit vector, tangent to streamline unit vector.

$i_b = i_z$ By right-hand-rule, normal to the above two unit vectors: The, “up” axis of a cylindrical system. Out-of-page axis here. No fluid dynamics along this axis.

$\frac{\delta}{\delta n} = -\frac{\delta}{\delta r}$ As the above relationship between streamline normal and cylindrical radial unit-vectors, this is the derivative relationship.

$\frac{\delta}{\delta s} = \frac{\delta}{r\delta\theta}$ Along-streamline derivative relationship

That last denominator might be confusing but it’s just as simple as spanning a distance ds with angular $d\theta$ change operating on a radius r .

If you now go back to the Fay PDF and accept Euler’s Equation in Cartesian coordinates on page 90-91 you can use the steps above to get confident with the cylindrical acceleration representation, also on page 90. From there you use the relationships just above to adjust the cylindrical representation to the streamline representation and you get to Fay’s page 105 below. We’re getting close!

Understanding, “lift” from Streamline Coordinates

The key equation for us is

$$-\frac{\delta p}{\delta n} = \rho \frac{V(n)^2}{R(n)}$$

I added the emphasis on V and R being likely functions of n, so integrating to estimate a pressure difference would be non-trivial. Sonin’s paper gives us a chance to solve the n-direction equation with models for Velocity and streamline radius as a function of n. See the paper to attempt this derivation for flow over a simple hill. I have not yet been successful arriving at equation (25) by using (23) and (24) to solve the differential equation (21) (I think it’s an integration-by-parts trick I am failing on).

As Fay and Sonin state, this is not a practical method for actually calculating a lift force. It is over-simplified, and we seldom have representations for V and R as a function of n over a range of s-vectors tangent to the flow over a wing. If we did, we’d still need to integrate them. However, this treatment tunes us into the physics of the problem. That’s our goal here: physical understanding, even if exact solutions elude us.

The approximation below from my 1995 Sonin lecture notes show us what we need to know in order to appreciate why our prop-speed-squared (ω^2) is our lift producer. The notes were taken here, which I remember because I recall daydreaming as I watched a guy mow the grass outside the lecture hall window one fine autumn day.

Lift lecture notes (must not have been staring out the window at this time)

Understanding Approximations for Lift

The n-direction equation tells us what we need to know to understand lift generated by airfoils (thrust from propellers: spinning airfoils). As Sonin's paper above states, "The n-direction equation states that when there is flow and the streamlines curve, the sum $p+\rho gz$ (which is constant when the fluid is static) *increases in the n-direction*, that is, as one moves *away* from the local center of curvature."

If we imagine integrating from the surface of the wing to far above the wing the streamlines way out at our approximated infinity would have a very large radius while the airspeed is V_∞

If we just assume airspeed is V_∞ everywhere (a crude approximation because we are speeding-it-up over the wing) but let the streamline radius get larger from top-surface radius R as we move away from the wing then we can estimate a low pressure on the top of the wing proportional to ρV_∞^2 as indicated in the lecture notes above.

Remember that if our propeller is spinning with angular velocity ω . At any point along the propeller where we could take a cross-section to sketch an airfoil diagram the "V" is just propeller rotational rate times the radius from the propeller axis to the location of our cross-section.

$$V_{cross-section} = \omega_{prop} \cdot r_{cross-section}$$

At this particular propeller cross-section we could estimate the free-stream velocity-squared as

$$V_\infty^2 = V_{cross-section}^2 = (\omega_{prop} \cdot r_{cross-section})^2 = \omega_{prop}^2 \cdot r_{cross-section}^2$$

There's our propeller ω^2 term!

We now see how propeller rotational speed squared (ω^2) is the variable responsible for, "lift" generated by a propeller blade. It is proportional to a simplified, assumed freestream velocity V^2 at any point along the blade where we take a cross-section as above. This is a highly simplified model, but it gets us to the essence of an airfoil.

Observe this on your next flight

Next time you land or take-off in an airplane observe the flaps that extend down-and-back from the rear of the wings. They create a shorter wing curvature radius and a larger wing surface. This increases lift. It is required because the plane is going much slower at take-off and landing than when cruising so the pressure difference across the same wing would be less at low speed, and it would be difficult to keep the plane, “up”.

So the modifiable airfoil extends the flaps, creates a larger surface over which the pressure difference acts, and puts more curvature into the airfoil to increase the pressure difference at low airspeed. This occurs at the expense of *drag* however. Given it is only needed at low speeds the flaps are retracted for cruising, where V_∞ is higher so we don’t need the curvature and extra area the flaps provide.

There remains that ρ term for air density, and we know it is higher in the troposphere down near the ground than up at 30,000 feet and higher. We’re going to ignore this though because our quad rotor is going to operate near the ground. Planes are more sensitive to, “stall” (failure of the lift from wings) at high altitude due to the rarefied (low density) air. Airspeed and angle-of-attack are critical factors, “up there”.

Conclusion

We now know that when our quad-rotor main controller asks our 4 motors for more-or-less speed many times per second it is the speed-squared (ω^2) that gives us lift! I know I wrote this before, and have taken some deeper-dive detours, but from here we should get back to the prop speed control loops and soon onto the quadrotor main platform stabilization control.

This post has been a great journey back to interesting topics I once studied. Just as through the last post, the opportunity to produce this write-up has forced me to (re)learn to more depth and understanding than I learned it the first time! I hope the references above and any gap-filling explanations I offer help convey the, "lift" phenomenon!