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# Modelling of the ETH Helicopter Laboratory Process

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# 1. Introduction

This report contains derivation of the dynamics of the ETH helicopter laboratory process, see Figure 1, using the Euler-Lagrange approach. The process is designed at the Automatic Control Laboratory at ETH in Zürich, see Mansour and Schaufelberger (1986) and Schaufelberger (1990). A description of the setup is found in Morari, M., W. Schaufelberger and A. Glattfelder (1995). The process is of MIMO type with nonlinear dynamics, and static input nonlinearities, as will be shown below. The present model is derived with the purpose of accurate simulation of the helicopter process. It may also prove helpful for nonlinear controller design. Identification of a linear model, and a linear controller design is presented in Åkesson, Gustafson and Johansson (1996).

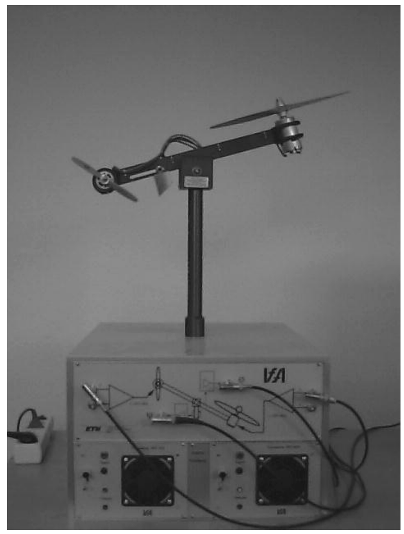

**Figure 1** The ETH helicopter laboratory process.

A schematic picture of the process is found in Figure 2. The helicopter consists of a vertical axle (**A**), on which a lever arm (**L**) is connected by a cylindric joint. The helicopter has two degrees of freedom: the rotation of the vertical axle (angle $\phi$) with respect to the fixed ground, and the pivoting of the lever arm (angle $\theta$) with respect to the vertical axle. Two rotors are mounted on the lever arm: $\mathbf{R_1}$ and $\mathbf{R_2}$, with the resultant aerodynamic forces giving rise to moments in the $\theta$ and $\phi$ directions respectively. The voltages $u_1$ and $u_2$ to the rotor motors are the inputs to the process. A weight is mounted at an adjustable position on the lever arm towards rotor $\mathbf{R_2}$.

# 2. Kinematics

It is assumed that the mass distribution on the lever arm is restricted to a straight line between the rotors, a distance $h$ from the pivot point. Denote by $O'$ an origo on this line. Let $[r_x(R), r_y(R), r_z(R)]$ denote a point $P$ on the lever arm parameterized in the distance $R$ from $O'$, expressed in an earth fixed reference system with origo in $O$ and oriented with $e_z$ along the

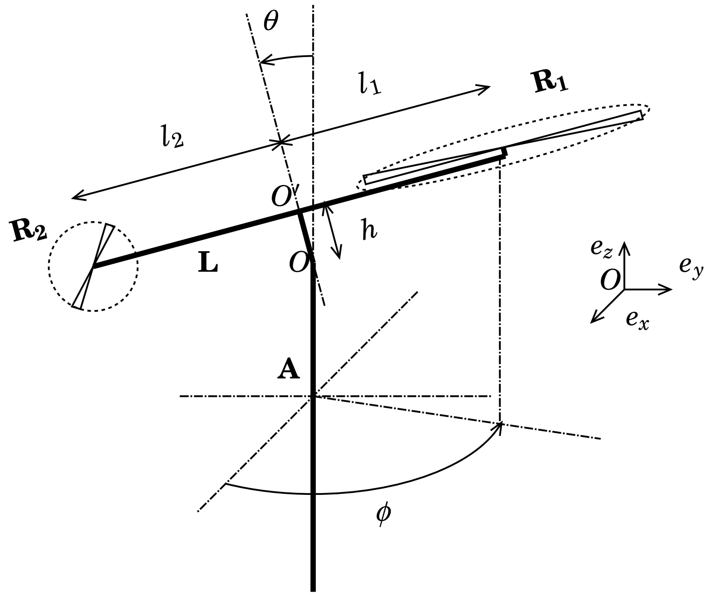


**Figure 2** Helicopter process configuration.

center axle. Then

$$
\begin{aligned}
r_x(R) &= R\cos\theta\cos\phi - h\sin\theta\cos\phi \\
r_y(R) &= R\cos\theta\sin\phi - h\sin\theta\sin\phi \\
r_z(R) &= R\sin\theta + h\cos\phi
\end{aligned}
\tag{1}
$$

The corresponding velocities are obtained from differentiation of (1) with respect to time:

$$
\begin{aligned}
v_x(R) &= -R\sin\theta\cos\phi\dot\theta - R\cos\theta\sin\phi\dot\phi - h\cos\theta\cos\phi\dot\theta + h\sin\theta\sin\phi\dot\phi \\
v_y(R) &= -R\sin\theta\sin\phi\dot\theta + R\cos\theta\cos\phi\dot\phi - h\cos\theta\sin\phi\dot\theta - h\sin\theta\cos\phi\dot\phi \\
v_z(R) &= R\cos\theta\dot\theta - h\sin\theta\dot\theta
\end{aligned}
\tag{2}
$$

The squared magnitude of the velocity of $P$ is then given by $v^2(R) = v_x^2(R) + v_y^2(R) + v_z^2(R)$:

$$
v^2(R) = R^2\left(\dot\theta^2 + \cos^2\theta\dot\phi^2\right) + h^2\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right) - 2hR\cos\theta\sin\theta\dot\phi^2 \tag{3}
$$

## 3. Energy expressions

The kinetic and potential energies are derived from

$$
T = \frac{1}{2}\int v^2(R)\,dm(R) \tag{4}
$$

$$
V = g\int r_z(R)\,dm(R) \tag{5}
$$

where $g$ is the acceleration of gravity. With (3) inserted this yields for lever arm

$$T_{\mathbf{L}} = \frac{1}{2}\left[\dot{\theta}^2 + \cos^2\theta\dot{\phi}^2\right] J_{\mathbf{L}} - h\cos\theta\sin\theta\dot{\phi}^2 ml_c + \frac{1}{2}h^2\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] m \tag{6a}$$

$$V_{\mathbf{L}} = mg\sin\theta l_c + mgh\cos\theta \tag{6b}$$

and for the axle

$$T_{\mathbf{A}} = \frac{1}{2}J_{\mathbf{A}}\dot{\phi}^2 \tag{6c}$$

$$V_{\mathbf{A}} = 0 \tag{6d}$$

where the inertia of the lever arm $J_{\mathbf{L}} \triangleq \int R^2\, dm(R)$, the center of gravity of the lever arm $l_c \triangleq \frac{1}{m}\int R\, dm(R)$, the lever arm mass $m \triangleq \int dm(R)$, and the center axle inertia $J_{\mathbf{A}}$ have been introduced. The total kinetic and potential energies are

$$T = T_{\mathbf{L}} + T_{\mathbf{A}} \tag{7}$$

$$V = V_{\mathbf{L}} + V_{\mathbf{A}} \tag{8}$$

## 4. Equations of motion

Forming the Lagrangian

$$L = T - V \tag{9}$$

the equations of motion are given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} &= \tau_\phi \\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} &= \tau_\theta \end{aligned} \tag{10}$$

Inserting (6) in (10) gives

$$\left[-2\cos\theta\sin\theta\dot{\phi}\dot{\theta} + \cos^2\theta\ddot{\phi}\right] J_{\mathbf{L}} + 2h\left[\left(\sin^2\theta - \cos^2\theta\right)\dot{\phi}\dot{\theta} - \cos\theta\sin\theta\ddot{\phi}\right] ml_c + h^2\left[2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + \sin^2\theta\ddot{\phi}\right] m + J_{\mathbf{A}}\ddot{\phi} = \tau_\phi \tag{11a}$$

$$\left[\ddot{\theta} + \cos\theta\sin\theta\dot{\phi}^2\right] J_{\mathbf{L}} + \left[h\left(-\sin^2\theta + \cos^2\theta\right)\dot{\phi}^2 + g\cos\theta\right] ml_c + \left[h^2\ddot{\theta} - h^2\sin\theta\cos\theta\dot{\phi}^2 - gh\sin\theta\right] m = \tau_\theta \tag{11b}$$

These equations may be expressed on matrix form as

$$D(\phi,\theta)\begin{bmatrix}\ddot{\phi}\\ \ddot{\theta}\end{bmatrix} + C(\phi,\theta,\dot{\phi},\dot{\theta})\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\end{bmatrix} + g(\phi,\theta) = \tau \tag{12}$$

The fundamental property $N(\phi,\theta,\dot{\phi},\dot{\theta}) = \dot{D}(\phi,\theta) - 2C(\phi,\theta,\dot{\phi},\dot{\theta})$ is fulfilled with the skew symmetric matrix $N(\phi,\theta,\dot{\phi},\dot{\theta})$. The matrices $D(\cdot)$, $C(\cdot)$, $g(\cdot)$, and $N(\cdot)$ are defined as in Figure 3.

$$D(\phi,\theta) \triangleq \begin{bmatrix} \cos^2\theta J_{\mathbf{L}} - 2h\cos\theta\sin\theta m l_c + h^2\sin^2\theta m + J_{\mathbf{A}} & 0 \\ 0 & J_{\mathbf{L}} + h^2 m \end{bmatrix}$$

$$C(\phi,\theta,\dot\phi,\dot\theta) \triangleq$$
$$\begin{bmatrix} -\cos\theta\sin\theta\dot\theta J_{\mathbf{L}} + h\left(\sin^2\theta - \cos^2\theta\right)\dot\theta m l_c + h^2\sin\theta\cos\theta\dot\theta m & -\cos\theta\sin\theta\dot\phi J_{\mathbf{L}} + h\left(\sin^2\theta - \cos^2\theta\right)\dot\phi m l_c + h^2\sin\theta\cos\theta\dot\phi m \\ \cos\theta\sin\theta\dot\phi J_{\mathbf{L}} + h\left(-\sin^2\theta + \cos^2\theta\right)\dot\phi m l_c - h^2\sin\theta\cos\theta\dot\phi m & 0 \end{bmatrix}$$

$$g(\phi,\theta) \triangleq \begin{bmatrix} o \\ g\cos\theta m l_c - mgh\sin\theta \end{bmatrix}$$

$$N(\phi,\theta,\dot\phi,\dot\theta) \triangleq$$
$$\begin{bmatrix} 0 & 2\cos\theta\sin\theta\dot\phi J_{\mathbf{L}} - 2h\left(\sin^2\theta - \cos^2\theta\right)\dot\phi m l_c - 2h^2\sin\theta\cos\theta\dot\phi m \\ -2\cos\theta\sin\theta\dot\phi J_{\mathbf{L}} - 2h\left(-\sin^2\theta + \cos^2\theta\right)\dot\phi m l_c + 2h^2\sin\theta\cos\theta\dot\phi m & 0 \end{bmatrix}$$

**Figure 3** Matrices for Equation (12).

## 5. Rotors

The rotors are driven by DC-motors without current-control. The motor operation is described by

$$L_a \frac{d}{dt} i_a = -R_a i_a - k\omega + u \tag{13a}$$

$$J \frac{d}{dt} \omega = T_d - T_L \tag{13b}$$

where $R_a$ and $L_a$ are the rotor-circuit resistance and inductance respectively, and $k$ the motor constant. The driving moment is described by

$$T_d = k i_a \tag{14}$$

and the motor load is described by

$$T_L = D\omega|\omega| \tag{15}$$

where $D$ is the aerodynamic torque coefficient, according to propeller Blade Element Theory, see e.g. Weick (1926) or, if preferred, a modern textbook on theory of flight. Combining (13–15) in steady-state yields

$$u_0 = \frac{R_a D}{k} \omega_0 |\omega_0| + k\omega_0 \approx k\omega_0 \tag{16}$$

The approximation is found valid by examining experimental results in Morari *et al.* (1995). The second order motor dynamics may then be approximated by first order dynamics as

$$\mathbf{R_1} : \frac{d}{dt}\omega_1 = -\frac{1}{T_1}\omega_1 + \frac{1}{k_1 T_1} u_1 \tag{17a}$$

$$\mathbf{R_2} : \frac{d}{dt}\omega_2 = -\frac{1}{T_2}\omega_2 + \frac{1}{k_2 T_2} u_2 \tag{17b}$$

where $T_1$ and $T_2$ are the time constants of the motors.

The resulting aerodynamic drag forces are given by $F_1 = C_1\omega_1|\omega_1|$ and $F_2 = C_2\omega_2|\omega_2|$ with $C_1$ and $C_2$ being aerodynamics drag coeffients (Weick 1926). Each rotor affects the helicopter with a moment resulting from the aerodynamic force, and with a moment that is the reaction moment from the driving torque of the rotor motor. (The sign of the reaction moments depend on the configuration of the rotor blades.) Gyroscopic effects of the rotors are assumed to be small and are neglected. (Any gyroscopic moments resulting from the rotor rotations would mainly include components perpendicular to the $\phi$ and $\theta$ rotation axis.) For $\mathbf{R_1}$:

$$\tau_{1,\phi} = D_1\omega_1|\omega_1|\cos\theta \tag{18a}$$

$$\tau_{1,\theta} = l_1 C_1 \omega_1 |\omega_1| \tag{18b}$$

and for $\mathbf{R_2}$:

$$\tau_{2,\phi} = l_2 \cos\theta C_2 \omega_2|\omega_2| \tag{19a}$$

$$\tau_{2,\theta} = D_2\omega_2|\omega_2| \tag{19b}$$

These moments are combined to form

$$\tau_\phi = \tau_{1,\phi} + \tau_{2,\phi} \tag{20a}$$

$$\tau_\theta = \tau_{1,\theta} + \tau_{2,\theta} \tag{20b}$$

## 6. Simulation model

The complete set of equations describing the helicopter process is given by (11) together with (17)–(20). Rewriting these on state-space form gives

$$
\begin{aligned}
\frac{d}{dt}\dot{\phi} &= \left[\cos^2\theta J_{\mathbf{L}} - 2h\cos\theta\sin\theta m l_c + h^2\sin^2\theta m + J_{\mathbf{A}}\right]^{-1} \\
&\quad\cdot\left[2\cos\theta\sin\theta\dot{\phi}\dot{\theta}J_{\mathbf{L}} - 2h\left(\sin^2\theta - \cos^2\theta\right)\dot{\phi}\dot{\theta}ml_c - 2h^2\sin\theta\cos\theta\dot{\phi}\dot{\theta}m\right. \\
&\quad\left. + D_1\omega_1|\omega_1|\cos\theta + l_2\cos\theta C_2\omega_2|\omega_2|\right] && \text{(21a)} \\
\frac{d}{dt}\phi &= \dot{\phi} && \text{(21b)} \\
\frac{d}{dt}\dot{\theta} &= \left[J_{\mathbf{L}} + h^2 m\right]^{-1}\cdot\left[-\cos\theta\sin\theta\dot{\phi}^2 J_{\mathbf{L}} - h\left(-\sin^2\theta + \cos^2\theta\right)\dot{\phi}^2 ml_c\right. \\
&\quad\left. -g\cos\theta ml_c + h^2\sin\theta\cos\theta\dot{\phi}^2 m + mgh\sin\theta + l_1C_1\omega_1|\omega_1| + D_2\omega_2|\omega_2|\right] && \text{(21c)} \\
\frac{d}{dt}\theta &= \dot{\theta} && \text{(21d)} \\
\frac{d}{dt}\omega_1 &= -\frac{1}{T_1}\omega_1 + \frac{1}{k_1T_1}u_1 && \text{(21e)} \\
\frac{d}{dt}\omega_2 &= -\frac{1}{T_2}\omega_2 + \frac{1}{k_2T_2}u_2 && \text{(21f)}
\end{aligned}
$$

## 7. Equilibrium points

Equations (11), (17)– (20) may be solved for stationary points $(\phi_0, \theta_0, u_{1,0}, u_{2,0})$ by setting $\ddot{\phi} = \ddot{\theta} = \dot{\phi} = \dot{\theta} = \dot{\omega}_1 = \dot{\omega}_2 \equiv 0$:

$$
\begin{aligned}
0 &= D_1\omega_{1,0}|\omega_{1,0}|\cos\theta + l_2\cos\theta C_2\omega_{2,0}|\omega_{2,0}| && \text{(22a)} \\
mg\left(l_c\cos\theta_0 - h\sin\theta_0\right) &= l_1C_1\omega_{1,0}|\omega_{1,0}| + D_2\omega_{2,0}|\omega_{2,0}| && \text{(22b)}
\end{aligned}
$$

For the unforced system with $\tau_\phi = \tau_\theta \equiv 0$ then $\phi = \phi_0$, $\theta = \theta_0$ are equilibrium points, with arbitrary $\phi_0$ and $\tan\theta_0 = l_c/h$. Since the $\tan^{-1}$ function is periodic there are infinitely many equilibrium points $\theta_0$. In particular there is one $\theta_0 \in [-\pi/2, \pi/2)$, and one $\theta_0 \in [-\pi, -\pi/2) \bigcup [\pi/2, \pi)$. Stability for the lever arm dynamics around $(\phi_0, \theta_0)$ may be investigated by regarding the resulting simplification and Taylor expansion of (11b):

$$
\begin{aligned}
\ddot{\theta} = \frac{mg}{J_{\mathbf{L}}}\left[h\sin\theta - l_c\cos\theta\right] &= \frac{mg}{J_{\mathbf{L}}}\left[h\cos\theta_0 + l_c\sin\theta_0\right]\delta\theta + O(\delta\theta^2) \\
&= \frac{mg\cos\theta_0}{hJ_{\mathbf{L}}}\left(h^2 + l_c^2\right)\delta\theta + O(\delta\theta^2) && \text{(23)}
\end{aligned}
$$

with $\delta\theta \triangleq \theta - \theta_0$. Thus the stationary point $(\phi_0, \theta_0)$ with $\theta_0 \in [-\pi/2, \pi/2)$ is stable for $h < 0$, and unstable for $h > 0$, and the stationary point with $\theta_0 \in [-\pi, -\pi/2) \bigcup [\pi/2, \pi)$ is unstable for $h < 0$, and stable for $h > 0$. The resulting dynamics is a pendulum equation.

## 8. Parameters

Parameters for a real helicopter process are presented in Morari *et al.* (1995). Geometric and interial parameters are presented directly. Motor and rotor properties are presented in graphs resulting from experiments. The corresponding parameters presented here are computed from the graphs.

| *Description* | *Parameter* | *Value* | *Unit* |
|---|---|---|---|
| Arm length to $\mathbf{R_1}$ | $l_1$ | 0.1995 | [m] |
| Arm length to $\mathbf{R_2}$ | $l_2$ | 0.1743 | [m] |
| Mass of lever arm bar | $m_l$ | 0.280 | [kg] |
| Pivot height | $h$ | 0.0298 | [m] |
| Mass of weight | $m_w$ | 0.158 | [kg] |
| Distance to weight (nominal[1]) | $l_w$ | 0.090 | [m] |
| Mass of rotor $\mathbf{R_1}$ | $m_1$ | 0.3792 | [kg] |
| Mass of rotor $\mathbf{R_2}$ | $m_2$ | 0.1739 | [kg] |
| Time constant for rotor $\mathbf{R_1}$ | $T_1$ | 1.1 | [s] |
| Time constant for rotor $\mathbf{R_2}$ | $T_2$ | 0.33 | [s] |
| Motor constant for rotor $\mathbf{R_1}$ | $k_1$ | $1.00 \cdot 10^{-2}$ | [Vs/rad] |
| Motor constant for rotor $\mathbf{R_2}$ | $k_2$ | $1.39 \cdot 10^{-2}$ | [Vs/rad] |
| Aerodynamic drag for rotor $\mathbf{R_1}$ | $C_1$ | $2.50 \cdot 10^{-5}$ | [$Ns^2/rad^2$] |
| Aerodynamic drag for rotor $\mathbf{R_2}$ | $C_2$ | $1.58 \cdot 10^{-6}$ | [$Ns^2/rad^2$] |
| Aerodynamic torque for rotor $\mathbf{R_1}$ | $D_1$ | $2.90 \cdot 10^{-7}$ | [$Nms^2/rad^2$] |
| Aerodynamic torque for rotor $\mathbf{R_2}$ | $D_2$ | $1.76 \cdot 10^{-7}$ | [$Nms^2/rad^2$] |

**Table 1** Helicopter model parameters.

The total mass of the lever arm is

$$m = m_l + m_1 + m_2 + m_w \tag{24}$$

The moment of inertia for the lever arm is the sum of the moment of inertia for the solid lever bar and for the point masses of the rotors and the weight:

$$J_{\mathbf{L}} = \frac{m_l}{3} \frac{l_1^3 + l_2^3}{l_1 + l_2} + m_1 l_1^2 + m_2 l_2^2 + m_w l_w^2 \tag{25}$$

The moment of inertia for the vertical axle may be neglected:

$$J_{\mathbf{A}} \approx 0 \tag{26}$$

The center of gravity is

$$l_c = \frac{m_l (l_1 - l_2) + m_1 l_1 - m_2 l_2 - m_w l_w}{m} \tag{27}$$

[1]May be varied in the range 0.0705 – 0.119 [m].

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