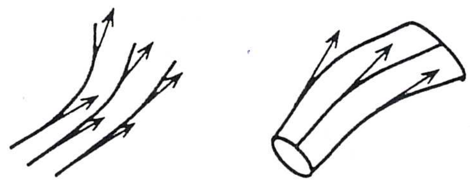

Figure 3.1: Sketch of streamlines (left) and a streamtube (right) in a flow field.

3.1.2 Streamlines and Pathlines

In colloquial usage, the word streamline connotes a smoothed flow, as in "streamlined body." In fluid mechanics, it has a precise meaning. A *streamline* is a line in a flow field that is everywhere tangent to the velocity vector $\mathbf{V}$ at each point along the streamline for any instant of time t. (See Figure 3.1.) We can think of the flow field being filled with streamlines, much as a pot is filled with cooked spaghetti. Like spaghetti, the streamlines never intersect each other because at any point there can be only one direction of the velocity. Later on we will show how to calculate the streamlines for some multidimensional flows. For the simple flow of Example 3.1, the streamlines are straight lines parallel to the x axis.

A *streamtube* is a surface in the flow formed from streamlines and closed upon itself to form a tube of variable cross-section, as shown in Figure 3.1. The use of the streamtube is primarily a conceptual one, helping us to visualize mentally how a flow field might be subdivided into streamtubes that entirely fill the flow field. Since the fluid does not cross through the surface of the stream tube, we may think of the stream tubes as flexible, moveable pipes containing the flow inside them.

Both streamlines and streamtubes are instantaneous snapshots of lines and surfaces in the flow field. As time progresses, these lines and surfaces will move to different locations in space unless the flow is steady; *i.e.*, does not depend upon time. In steady flow, any snapshot of the flow is identical to every other one and the streamlines and stream surfaces are fixed in space.

The useful line to define in the Lagrangian description of a flow field is the *pathline*, which is the path followed, over later times, of a particular particle identified at an initial time and location. We can observe a pathline in an experiment by marking a fluid particle with a puff of dye or smoke and taking a time–exposed picture of the marked fluid. (A common analog is the light path of auto headlights in a nightime time–exposed photograph.) A pathline is the trajectory of a single fluid particle.

If a flow is steady, then a streamline and pathline passing through the same point in space are identical because the velocity field depends only upon position and not time. Experimentally, this is convenient for the observation of steady streamlines because the set of pathlines formed by a steady stream (*i.e.*, a closely spaced series of puffs) of dye or smoke follow the same trajectory through space as the streamline. Even in unsteady flows, the use of dye or smoke markers can be helpful in visualizing the flow behavior despite the fact that the marked fluid does not denote either a streamline or a pathline.

3.1.3 The Material Derivative

The standard forms of Newton's law of motion and the laws of thermodynamics apply to a fixed mass of identified matter whose properties change as time progresses. The natural mode for expressing these laws is the Lagrangian description of motion because it directly describes the history of an identified particle. Since we use the Eulerian description for a moving fluid, we need to establish the Eulerian expression of the rate of change of any property of a fluid particle as it moves through the flow field. The time rate of change of a fluid property, as measured by an observer moving with the particle, is called the *material derivative* of that property.

As an example, consider the rate of change of density ρ of a fluid particle that is located at position $\mathbf{R}$ at time t. During the time interval dt, the particle moves an amount $d\mathbf{R} = \mathbf{V}\,dt$. The total increment $d\rho$ in density is the sum of the part due to the time increment dt and that due to the spatial increment $d\mathbf{R}$. Using Cartesian coordinates to express the amount of $d\rho$,

$$\begin{aligned}
d\rho &= \frac{\partial \rho}{\partial t}dt + \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz \\
&= \frac{\partial \rho}{\partial t}dt + \left(\frac{\partial \rho}{\partial x}\mathbf{i}_x + \frac{\partial \rho}{\partial y}\mathbf{i}_y + \frac{\partial \rho}{\partial z}\mathbf{i}_z\right) \cdot (\mathbf{i}_x dx + \mathbf{i}_y dy + \mathbf{i}_z dz) \\
&= \frac{\partial \rho}{\partial t}dt + \nabla\rho \cdot d\mathbf{R} = \frac{\partial \rho}{\partial t}dt + d\mathbf{R} \cdot \nabla\rho \\
&= \frac{\partial \rho}{\partial t}dt + \mathbf{V}dt \cdot \nabla\rho \\
&= \left(\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla\rho\right) dt \\
\frac{d\rho}{dt} &= \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right) \rho \qquad (3.1)
\end{aligned}$$

To emphasize that the material time derivative includes both spatial and time partial derivatives, and is not simply the partial time derivative, we will denote it by D/Dt:

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right) \qquad (3.2)$$

Note that D/Dt is a scalar operator so that the material derivative of a scalar variable, such as density ρ, for example, is a scalar quantity:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{V} \cdot \nabla)\rho \qquad (3.3)$$

Equation 3.3 may be expressed in terms of Cartesian coordinates as :

$$\begin{aligned}
\frac{D\rho}{Dt} &= \frac{\partial \rho}{\partial t} + \left[(u\mathbf{i}_x + v\mathbf{i}_y + w\mathbf{i}_z) \cdot (\mathbf{i}_x\frac{\partial}{\partial x} + \mathbf{i}_y\frac{\partial}{\partial y} + \mathbf{i}_z\frac{\partial}{\partial z})\right] \rho \\
&= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \qquad (3.4)
\end{aligned}$$

and in terms of cylindrical coordinates as:

$$\begin{aligned}
\frac{D\rho}{Dt} &= \frac{\partial \rho}{\partial t} + \left[(V_r\mathbf{i}_r + V_\theta\mathbf{i}_\theta + V_z\mathbf{i}_z) \cdot (\mathbf{i}_r\frac{\partial}{\partial r} + \mathbf{i}_\theta\frac{\partial}{r\partial \theta} + \mathbf{i}_z\frac{\partial}{\partial z})\right] \rho \\
&= \frac{\partial \rho}{\partial t} + V_r\frac{\partial \rho}{\partial r} + \frac{V_\theta}{r}\frac{\partial \rho}{\partial \theta} + V_z\frac{\partial \rho}{\partial z} \qquad (3.5)
\end{aligned}$$

Expressions for the material derivative of a vector property, such as the velocity $\mathbf{V}$, will be treated in section 4.2.

Example 3.2 A velocity field and density field in Cartesian space is given as:

$$\begin{aligned}
\mathbf{V} &= \frac{L}{t}\mathbf{i}_x \\
\rho &= Kte^{-x/L}
\end{aligned}$$

where L and K are constants having the dimensions of length and density ÷ time, respectively. Find the material derivative of the density ρ.

Solution Substituting into Equation 3.3,

$$\begin{aligned}
\frac{D\rho}{Dt} &= \frac{\partial}{\partial t}\left(Kte^{-x/L}\right) + \frac{L}{t}\frac{\partial}{\partial x}\left(Kte^{-x/L}\right) \\
&= Ke^{-x/L} + \frac{L}{t}\left(\frac{-Kt}{L}e^{-x/L}\right) \\
&= 0
\end{aligned}$$

Chapter 4

Inviscid Flow

In this chapter we consider a kind of flow, called *inviscid flow*, that occurs in special, although not uncommon, circumstances. In such flows the effect of fluid viscosity is so small as to be ignorable and the resultant flow is much easier to treat analytically than is the case when viscous effects cannot be ignored. More importantly, it is possible to determine readily significant properties of the flow, such as the pressure and velocity fields, or even to estimate them by use of simple algebraic relations. It is easy to develop physical intuition about how an inviscid flow behaves.

We first derive the vector differential equation of motion of an inviscid fluid, called *Euler's equation*. Then we find a scalar integral of this equation, called *Bernoulli's equation*, that provides an algebraic relation between pressure, velocity and position in the earth's gravitational field. While Bernoulli's equation doesn't tell us everything about the flow field, it may provide us with enough information to find what we need to know to solve a particular practical problem.

Later, in Chapter 6, we will consider the more difficult problem of how to describe a flow when viscous effects cannot be neglected. For the time being, by treating only inviscid flows we can develop some familiarity with fluid dynamical principles that will be helpful when we consider the more general case of viscous flows.

4.1 Criterion for Inviscid Flow

If a fluid were to have zero viscosity, then it could not sustain a shear stress and its flow would be inviscid exactly. But no fluid has zero viscosity.[1] For a flow to be regarded as inviscid, the effects of the shear stresses on the motion must be sufficiently small compared to other influences that they can be ignored as being negligible. A *necessary*, but not *sufficient*, condition for negligible viscous effects is that a dimensionless parameter characterizing the flow, called the *Reynolds number* and denoted by Re, is very large. For the steady flow of a fluid of density ρ and viscosity μ over (or through) an object of dimension L at a speed V, the Reynolds number is:

$$Re \equiv \frac{\rho V L}{\mu} = \frac{V L}{\nu} \tag{4.1}$$

where we have used Equation 1.6 to replace ρ/μ by $1/\nu$. Thus the necessary condition for inviscid flow is:

$$Re >> 1$$

If the Reynolds number of a flow is not large, then the flow is viscous and cannot be treated as an inviscid flow. However, it is possible that a large Reynolds number flow can be greatly affected by viscous effects under some circumstances, such as when the flow comes in contact with solid boundaries. We cannot always predict when such flows should be regarded as viscous and must be

[1] Liquid helium-4 at temperatures below 4.2 K flows without friction through small tubes and channels. This flow is a macroscopic quantum motion of the fluid and is not describable in its entirety by Euler's equation. Under these conditions helium-4 is called a *superfluid*.

guided by experimental observation and experience. For the time being, we will note in this chapter when a flow cannot be treated as inviscid.

4.2 Acceleration of a Fluid Particle

If we wish to express Newton's law of motion for a fluid particle in the form, *mass* × *acceleration* = *force*, we need first an expression for the acceleration of a fluid particle. Since the acceleration of a fluid particle is the time rate of change of its velocity **V**,

$$\textbf{acceleration} = \frac{D\mathbf{V}}{Dt} \tag{4.2}$$

where we have used the material time derivative D/Dt of Equation 3.2 because we need the time rate of change following the fluid particle. In Cartesian coordinates, the acceleration $D\mathbf{V}/Dt$ may be written in component form as:

$$\begin{aligned} \frac{D\mathbf{V}}{Dt} &= \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} \\ &= \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\mathbf{i}_x \\ &+ \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\mathbf{i}_y \\ &+ \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\mathbf{i}_z \end{aligned} \tag{4.3}$$

while in cylindrical coordinates the acceleration becomes:

$$\begin{aligned} \frac{D\mathbf{V}}{Dt} &= \left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right)\mathbf{i}_r \\ &+ \left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right)\mathbf{i}_\theta \\ &+ \left(\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right)\mathbf{i}_z \end{aligned} \tag{4.4}$$

Each of the three components of the acceleration vector requires one time derivative and three spatial derivatives of a velocity component in its expression, or a total of twelve derivatives needed for determining the acceleration of a fluid particle. When expressed in cylindrical coordinates (Equation 4.4), there are two additional, non-derivative terms: the centrifugal acceleration $-V_\theta^2/r$ in the radial direction and the Coriolis acceleration $V_r V_\theta/r$ in the tangential direction. Since it would be awkward to write out the complete form of the acceleration in terms of these derivatives every time we need to use it in an equation, we shall use the shorthand notation $D\mathbf{V}/Dt$ or $\partial\mathbf{V}/\partial t + (\mathbf{V}\cdot\nabla)\mathbf{V}$ to indicate the acceleration of a fluid particle.

Example 4.1 The velocity field of a steady incompressible inviscid flow, expressed in cylindrical coordinates, is:

$$V_r = \frac{k_1}{r};\ V_\theta = \frac{k_2}{r};\ V_z = 0$$

where k_1 and k_2 are constants having the dimension of *velocity* × *length.* Derive expressions for the components of acceleration in the radial, tangential and axial directions.

Solution Substituting the velocity components into Equation 4.4,

$$\frac{D\mathbf{V}}{Dt}\cdot\mathbf{i}_r = 0 + \frac{k_1}{r}\left(-\frac{k_1}{r^2}\right) + 0 + 0 - \frac{1}{r}\left(\frac{k_2}{r}\right)^2 = -\frac{k_1^2 + k_2^2}{r^3}$$

$$\frac{D\mathbf{V}}{Dt}\cdot\mathbf{i}_\theta = 0+\frac{k_1}{r}\left(-\frac{k_2}{r^2}\right)+0+0+\frac{k_1k_2}{r^3}=0$$
$$\frac{D\mathbf{V}}{Dt}\cdot\mathbf{i}_z = 0$$

4.3 Euler's Equation

We are now equipped to write Newton's law of motion for a fluid particle. Select a volume element $\delta\mathcal{V}$ of fluid having a mass $\rho\,\delta\mathcal{V}$. Because $-\nabla p$ is the pressure force per unit volume of fluid, this volume element is subject to a pressure force $(-\nabla p)\,\delta\mathcal{V}$. It is also acted upon by a gravity force $(\rho\,\delta\mathcal{V})\,\mathbf{g}$. Equating the product of mass $\rho\,\delta\mathcal{V}$ times the acceleration of a fluid particle to the sum of the pressure force $(-\nabla p)\,\delta\mathcal{V}$ and the gravity force $(\rho\,\delta\mathcal{V})\,\mathbf{g}$ acting on the particle, we write the equation of motion as:

Ma = ΣF

$$(\rho\,\delta\mathcal{V})\left(\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right)=(-\nabla p)\,\delta\mathcal{V}+(\rho\,\delta\mathcal{V})\,\mathbf{g}$$

Dividing by $\rho\,\delta\mathcal{V}$, we obtain *Euler's equation:*[2]

$$\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}=-\frac{1}{\rho}\nabla p+\mathbf{g} \tag{4.5}$$

The left side of Euler's equation is the acceleration of a fluid particle while the right side is the sum of the forces per unit mass of fluid. Note that the fluid density appears only in the denominator of the pressure force term. For a given amount of acceleration, a high density liquid fluid particle requires a much greater pressure gradient ∇p than does a low density gas particle. On the other hand, a droplet of water and one of mercury would fall freely in a vacuum ($\nabla p = 0$) with the same acceleration $\mathbf{g}$ despite their different densities. In the absence of any motion ($\mathbf{V}=0$), Euler's equation reduces to the equation for static equilibrium, Equation 2.6.

For use in working problems and examples utilizing Cartesian coordinates, we write here the x, y and z components of Euler's equation, found by evaluating the scalar product of Equation 4.5 times $\mathbf{i}_x, \mathbf{i}_y$ and $\mathbf{i}_z$, respectively:

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\mathbf{g}\cdot\mathbf{i}_x\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\mathbf{g}\cdot\mathbf{i}_y\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\mathbf{g}\cdot\mathbf{i}_z
\end{aligned} \tag{4.6}$$

Example 4.2 A steady inviscid incompressible flow has a velocity field:

$$u = fx;\ v=-fy;\ w=0$$

where f is a constant having the dimensions of s^{-1}. Derive an expression for the pressure field $p\{x,y,z\}$ if the pressure $p\{0,0,0)\} = p_0$ and $\mathbf{g}=-g\mathbf{i}_z$.

[2] Leonhard Euler (1707-1783) was one of the most prolific mathematicians of all time. He made many contributions to mechanics, dynamics and hydrodynamics. He proved that Bernoulli's equation is an integral of Euler's equation.

Solution Substituting the values of the velocity components into Euler's equation 4.6, we find:

$$fx\frac{\partial(fx)}{\partial x} - fy\frac{\partial(fx)}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}; \quad or\ f^2x = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$fx\frac{\partial(-fy)}{\partial x} - fy\frac{\partial(-fy)}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}; \quad or\ f^2y = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Integrating the first of these equations,

$$\frac{p}{\rho} = -\frac{(fx)^2}{2} + h\{y, z\}$$

and substituting into the second and integrating:

$$f^2y = -\frac{\partial h}{\partial y}$$

$$h = -\frac{(fy)^2}{2} + k\{z\}$$

The expression for p now becomes:

$$\frac{p}{\rho} = -\frac{f^2}{2}(x^2 + y^2) + k\{z\}$$

Substituting this expression for p into the third of Euler's equations and integrating:

$$0 = -\frac{\partial k}{\partial z} - g$$

$$k\{z\} = -gz + c$$

Substituting $k\{z\}$ into the expression for p gives:

$$\frac{p}{\rho} = -\frac{f^2}{2}(x^2 + y^2) - gz + c$$

Finally, we choose $c = p_0/\rho$ so as to satisfy the condition that $p\{0, 0, 0\} = p_0$:

$$p = p_0 - \rho gz - \rho\frac{f^2}{2}(x^2 + y^2)$$

4.3.1 Constant–Density Flow

If the fluid density is constant throughout the flow field and unvarying with time – an instance of incompressible flow – it is possible to simplify the form of Euler's equation by defining a new independent variable p^*:

$$p^* \equiv p - \rho\mathbf{g}\cdot\mathbf{R} \tag{4.7}$$

so that Euler's equation 4.5 takes the form:

$$\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla(p^* + \rho\mathbf{g}\cdot\mathbf{R}) + \mathbf{g}$$

$$= -\frac{1}{\rho}\nabla p^* \qquad (if\ \ \nabla\rho = 0) \tag{4.8}$$

4.4 Bernoulli's Equation

To solve an inviscid fluid flow problem utilizing Cartesian coordinates, we must integrate Euler's equation to find the four dependent scalar variables u, v, w and p as functions of the independent variables x, y, z and t (assuming ρ is a constant). Since we need four scalar equations to find the four dependent variables, we must append the equation of mass conservation of an incompressible fluid, Equation 3.17,

$$\nabla \cdot \mathbf{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (4.9)$$

to the three scalar components of Euler's equation, Equation 4.6. Solving such a complex set of equations is indeed a formidable problem. No general integral of these equations has yet been found. Even a numerical solution for an arbitrary three-dimensional unsteady motion requires the use of a supercomputer. But if we limit our attention to flows of simple geometry and initial and boundary conditions, we can find some analytical solutions that are practically useful. Examples are given in Chapter 11.

It was the genius of Bernoulli[3] to have derived what subsequently proved to be a single scalar integral of Euler's equation, one that applies to any inviscid flow provided the fluid density does not vary arbitrarily, but only in a prescribed manner. This integral, called *Bernoulli's equation*, is often directly applicable to many engineering problems, providing useful, although not complete, information about the fluid flow. (For a complete description of the fluid flow, we would need four scalar integrals of Euler's equation and the mass conservation equation.)

To derive Bernoulli's equation, we begin by using the vector identity of Equation 1.40 to replace the term $(\mathbf{V}\cdot\nabla)\mathbf{V}$ in Equation 4.5 and rearranging the terms to obtain the following form of Euler's equation:

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) + \frac{1}{\rho}\nabla p - \mathbf{g} = \mathbf{V}\times(\nabla\times\mathbf{V})$$

Next we integrate this form of Euler's equation along a line C in space, whose element of length is dc, by forming the scalar product of Euler's equation and dc, then integrating between the points 1 and 2 along the line C:

$$\int_1^2 \frac{\partial \mathbf{V}}{\partial t}\cdot d\mathbf{c} + \int_1^2 \nabla\left(\frac{V^2}{2}\right)\cdot d\mathbf{c} + \int_1^2 \left(\frac{1}{\rho}\nabla p\right)\cdot d\mathbf{c} - \int_1^2 \mathbf{g}\cdot d\mathbf{c} = \int_1^2 \mathbf{V}\times(\nabla\times\mathbf{V})\cdot d\mathbf{c} \qquad (4.10)$$

Two of the terms in this equation can be integrated directly using Equation 1.46 and Equation 2.7:

$$\begin{aligned}\int_1^2 \nabla\left(\frac{V^2}{2}\right)\cdot d\mathbf{c} &= \frac{V_2^2}{2} - \frac{V_1^2}{2}\\ \int_1^2 \mathbf{g}\cdot d\mathbf{c} &= \int_1^2 \nabla(\mathbf{g}\cdot\mathbf{R})\cdot d\mathbf{c}\\ &= \mathbf{g}\cdot\mathbf{R}_2 - \mathbf{g}\cdot\mathbf{R}_1\end{aligned}$$

To integrate other terms, choose the line C to be a streamline; *i.e.*, dc is parallel to $\mathbf{V}$ at each point along the line. To emphasize this choice, we denote the streamline element by ds. By this choice, the integral on the right side of Equation 4.10 is zero because its integrand is perpendicular to $\mathbf{V}$ and the scalar product of the integrand with ds is identically zero:

$$\int_1^2 \mathbf{V}\times(\nabla\times\mathbf{V})\cdot d\mathbf{s} = 0 \qquad (\textit{if}\ d\mathbf{s}\times\mathbf{V} = 0) \qquad (4.11)$$

where the condition that ds is parallel to $\mathbf{V}$ can be expressed as $d\mathbf{s}\times\mathbf{V} = 0$. The pressure gradient integral can be evaluated easily if the density does not change along the streamline: [4]

$$\begin{aligned}\int_1^2 \left(\frac{1}{\rho}\nabla p\right)\cdot d\mathbf{s} &= \frac{1}{\rho}\int_1^2 \nabla p\cdot d\mathbf{s}\\ &= \frac{1}{\rho}(p_2 - p_1) \qquad (\textit{if}\ \mathbf{V}\cdot(\nabla\rho) = 0) \qquad (4.12)\end{aligned}$$

where the constancy of density along a streamline is ensured by the condition that the density gradient $\nabla\rho$ is perpendicular to $\mathbf{V}$, or $\mathbf{V}\cdot(\nabla\rho) = 0$. Inserting these values for the integrals into Equation 4.10,

$$\int_1^2 \frac{\partial \mathbf{V}}{\partial t}\cdot d\mathbf{s} + \left(\frac{\mathbf{V}_2^2}{2} + \frac{p_2}{\rho} - \mathbf{g}\cdot\mathbf{R_2}\right) - \left(\frac{\mathbf{V}_1^2}{2} + \frac{p_1}{\rho} - \mathbf{g}\cdot\mathbf{R_1}\right) = 0$$

[3] Daniel Bernoulli (1700-1782) was both a mathematician, hydrodynamicist and physician. His *Hydrodynamica* (1738) explains his "equation" but does not specifically derive it from first principles. The derivation was subsequently given by Euler.

[4] It is not sufficient that the flow be incompressible ($D\rho/Dt = 0$) for the density to be constant along a streamline, unless the flow is also steady. However, it is sufficient, but not necessary, for the density to be constant everywhere in the flow field to satisfy its constancy along a streamline.

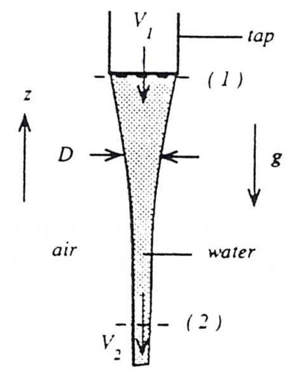

Figure 4.1: Water leaving a tap increases in speed as it falls downward through the stationary air, as described by Bernoulli's equation.

$$(if\ \mathbf{V}\cdot(\nabla\rho) = 0;\ d\mathbf{s}\times\mathbf{V} = 0) \tag{4.13}$$

This is the form of *Bernoulli's equation* for the case of constant density along a streamline.

In unsteady flow, the integrand of the first term of Bernoulli's equation, Equation 4.13, must be known at all points along the instantaneous streamline in order for the integral to be evaluated. If the flow is such that the streamlines do not change with time but that the velocity $\mathbf{V}$ does, then $\partial\mathbf{V}/\partial t$ has the direction of $\mathbf{V}$ and ds and the integral may be easy to evaluate.

In a steady flow the first term of Equation 4.13 is absent and the sum, $\mathbf{V}^2/2 + p/\rho - \mathbf{g}\cdot\mathbf{R}$, has the same value at all points along the same streamline, but not necessarily the same value as points along a different streamline.

In our study of hydrostatics, we made use of a convention for a Cartesian coordinate system that the direction of the z-axis is vertical and opposite to the direction of $\mathbf{g}$. In this convention, $\mathbf{g}\cdot\mathbf{R} = -gz$ and Bernoulli's equation takes the form:

$$\int_1^2 \frac{\partial\mathbf{V}}{\partial t}\cdot ds + \left(\frac{\mathbf{V}_2^2}{2} + \frac{p_2}{\rho} + gz_2\right) - \left(\frac{\mathbf{V}_1^2}{2} + \frac{p_1}{\rho} + gz_1\right) = 0$$

$$(if\ \mathbf{V}\cdot(\nabla\rho) = 0;\ ds\times\mathbf{V} = 0;\ \mathbf{g} = -g\mathbf{i}_z) \tag{4.14}$$

This form of Bernoulli's equation is convenient for working most problems.

4.4.1 Applications of Bernoulli's Equation

Bernoulli's equation is very useful in enabling us to understand the behavior of many engineering flows. In this section we give examples of both steady and unsteady flows of incompressible fluids that demonstrate the application of Bernoulli's equation to the flow of fluids.

Fluid Streams

One of the simplest inviscid fluid flows is that of a stream of fluid (such as water) flowing through a stationary fluid with which it does not mix (such as air). Consider the case of water flowing from a tap, as illustrated in Figure 4.1. The water leaves the tap with a speed V_1 as a circular stream of diameter D_1. As it falls, it speeds up and contracts in diameter. Ultimately, the stream becomes so thin that surface tension forces break it up into droplets. But before this happens, the flow can be described by applying Bernoulli's equation 4.14 to the central streamline of the water stream:

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

On the central streamline the pressure of the water will be the same as that in the atmosphere at the same height z because the radial acceleration of the water stream is negligible. (We also

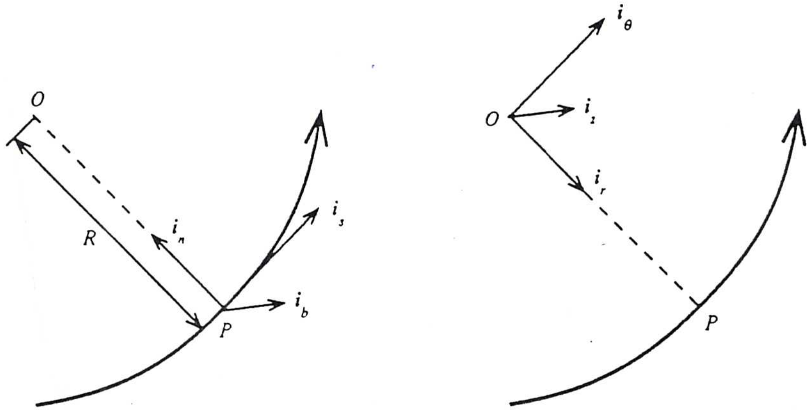

Figure 4.7: Unit vectors for streamline coordinates at a point P on a streamline whose center of curvature is O.

4.5 Euler's Equation in Streamline Coordinates

It is sometimes convenient to choose an orthogonal coordinate system whose local directions are defined by a streamline of the flow. Called streamline coordinates, the three mutually perpendicular directions at a point in the flow are determined by the directions of the tangent, normal and binormal to the streamline passing through the point. This is illustrated in Figure 4.7 for a point P on a streamline, where the unit vectors lying in these three directions are labeled $\mathbf{i}_s$, $\mathbf{i}_n$ and $\mathbf{i}_b$, respectively. The unit normal $\mathbf{i}_n$ points in the direction of the center of curvature O of the streamline at the point P, the distance OP being the radius of curvature R. To develop Euler's equation for this coordinate system, we will embed a cylindrical coordinate system with center at O and with the z-axis in the direction of the binormal $\mathbf{i}_b$. The point P lies at a radius $r = R$. The unit vectors of the streamline and cylindrical coordinates are related by $\mathbf{i}_r = -\mathbf{i}_n$, $\mathbf{i}_\theta = \mathbf{i}_s$, $\mathbf{i}_z = \mathbf{i}_b$, the magnitude of the components of the velocity vector by $V_r = V_n$, $V_\theta = V$, $V_z = V_b$, and the spatial derivatives by $\partial/\partial r = -\partial/\partial n$, $\partial/r\partial\theta = \partial/\partial s$, $\partial/\partial z = \partial/\partial b$. Substituting these values into Equation 4.4, and noting that $V_n = 0$ and $V_b = 0$ at any point on a streamline, the acceleration of a fluid particle in streamline coordinates is: [10]

$$\frac{D\mathbf{V}}{Dt} = \left(-\frac{\partial V_n}{\partial t} + \frac{V^2}{R}\right)\mathbf{i}_n + \left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}\right)\mathbf{i}_s + \left(\frac{\partial V_b}{\partial t}\right)\mathbf{i}_b \tag{4.18}$$

In the case of *steady flow* in streamline coordinates, Euler's equation has an especially simple form:

$$\begin{aligned}
\left(\frac{V^2}{R}\right) &= -\frac{1}{\rho}\frac{\partial p}{\partial n} + \mathbf{g}\cdot\mathbf{i}_n = -\frac{1}{\rho}\frac{\partial p^*}{\partial n} \\
V\frac{\partial V}{\partial s} &= -\frac{1}{\rho}\frac{\partial p}{\partial s} + \mathbf{g}\cdot\mathbf{i}_s = -\frac{1}{\rho}\frac{\partial p^*}{\partial s} \\
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial b} + \mathbf{g}\cdot\mathbf{i}_b = -\frac{1}{\rho}\frac{\partial p^*}{\partial b} \qquad (\textit{steady flow})
\end{aligned} \tag{4.19}$$

There are several aspects of these equations that deserve notice. The first of the equations of 4.19, which expresses the motion in the normal direction, shows that the net force in the direction $\mathbf{i}_n$ causes the centrifugal acceleration V^2/R. The second equation, for motion along the streamline, can be integrated to give Bernoulli's equation for steady flow. The third equation of motion in the direction of the binormal gives a hydrostatic pressure distribution in this direction because the acceleration in this direction is zero.

[10] Although V_n and V_b are zero on the instantaneous streamline passing through P, their time derivatives are not zero in general, unless the flow is steady. If we expand any component of the velocity in a Taylor series in time, we can see that the first and higher derivatives in time are not necessarily zero when the first term is zero.

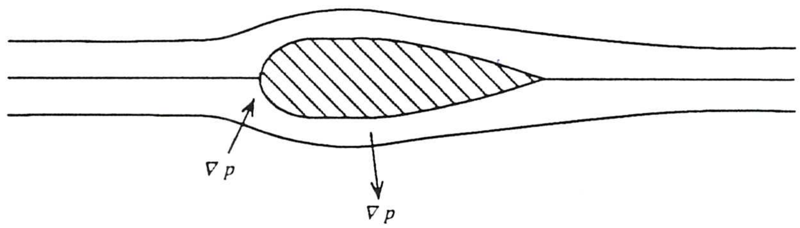

Figure 4.8: Sketch of streamlines near a body showing how the direction of the pressure gradient depends upon the curvature of the streamline.

The equation of motion in the normal direction enables us to determine how the pressure varies in a steady flow if we know the shape of the streamlines. If gravity has no component in the normal direction ($\mathbf{g} \cdot \mathbf{i}_n = 0$), then *the pressure decreases in the direction of the center of curvature of the streamline.* For example, consider the steady flow past a streamlined body sketched in Figure 4.8. At the front of the body, the streamlines curve away from the body and the pressure there must be higher than the uniform pressure far from the body. On the other hand, at the side of the body the streamlines follow the convex shape of the body and the pressure at the side must be less than that far away. While such a description of the pressure distribution in this flow is qualitatively correct, a quantitative calculation of the pressure in the flow cannot be obtained easily from the streamline form of Euler's equation, Equation 4.19, except for very simple flows.

4.6 Inviscid Flow in Noninertial Reference Frames

In analyzing fluid flows it is sometimes convenient to use a coordinate system tied to a reference frame that is not inertial; *i.e.*, one in which the acceleration measured by an observer in the noninertial reference frame is different from that measured by an observer in the inertial reference frame. Typical examples of such noninertial reference frames would be a reference frame fixed in a booster rocket that is accelerating upward from its launching pad or a rotating reference frame fixed to the rotor of a turbomachine. In such instances, the expressions for Euler's and Bernoulli's equations need to be modified to take into account the motion of the noninertial reference frame.

We commonly regard the laboratory reference frame as an inertial reference frame when describing fluid flow in laboratory experiments. However, because the earth rotates about its axis, the laboratory reference frame is not strictly inertial but can be regarded as such for flows having small length and time scales typical of engineering systems. For large scale flows that change slowly with time, such as that of the atmosphere or the ocean, it is necessary to take into account the earth's rotational speed when using a reference frame fixed to the earth. If the angular velocity of a rotating system is sufficiently high, the use of noninertial reference frames may require corrections that are important even for laboratory scale flows.

In this section we develop the modifications to the equations of inviscid flow that are required in the use of noninertial reference frames for two simple cases. The first of these is that of a translating (but not rotating) reference frame that is accelerating. The second is that of a reference frame that is rotating at a steady angular speed about an axis whose direction is fixed but is otherwise not being translated with respect to the inertial reference frame. These two examples suffice to cover most applications of engineering importance.

4.6.1 Translating, Accelerating Reference Frame

Consider a noninertial reference frame that is translating with respect to the inertial reference frame. (Translation means that the coordinate system axes do not change direction in inertial space.) Denote the position and velocity of a fluid particle in the noninertial reference frame by $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{V}}$, respectively, to distinguish them from the position $\mathbf{R}$ and velocity $\mathbf{V}$ in the inertial reference frame. (Because we are dealing with nonrelativistic velocities, time t will be the same in