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1

Longitudinal Dynamics

1-1 INTRODUCTION

To obtain the transfer function of the aircraft it is first necessary to obtain the equations of motion for the aircraft. The equations of motion are derived by applying Newton's laws of motion, which relate the summation of the external forces and moments to the linear and angular accelerations of the system or body. To make this application, certain assumptions must be made and an axis system defined.

The center of the axis system is, by definition, located at the center of gravity of the aircraft. In general, the axis system is fixed to the aircraft and rotates with it. Such a set of axes is referred to as "body axes." It is not necessary to use such an axis system; an axis system could be fixed, for example, to the air mass, and the aircraft could rotate with respect to it. However, for the purposes of this text the axis system is taken as fixed to the aircraft.

The axis is taken with OX forward, OY out the right wing, and OZ downward as seen by the pilot, to form a right-handed axis system (see Figure 1-2, p. 15). Most aircraft are symmetrical with reference to a vertical plane aligned with the longitudinal axis of the aircraft. Thus, if the OX and OZ axes lie in this plane, the products of inertia J_{xy} and J_{yz} are zero. This result leads to the first assumption:

1. *The axes OX and OZ lie in the plane of symmetry of the aircraft, and J_{xy} and J_{yz} are equal to zero.* At this time, the exact direction of OX is not specified, but in general it is not along a principal axis; hence $J_{zx} \neq 0$.

1-2 THE MEANING OF VELOCITIES IN A MOVING AXIS SYSTEM

Very often a student has difficulty understanding what is meant by the velocity of a body with respect to an axis system that is moving with the body. How can there be any relative velocity in this situation? Statements about the velocity along the OX axis refer to the *component of velocity with respect to inertial space* taken along the instantaneous direction of the OX axis. At any instant, the aircraft has some resultant velocity vector with respect to inertial space. This vector is resolved into the instantaneous aircraft axes to obtain the velocity components U, V, and W. This resolution also applies to the angular velocity. Resolve the instantaneous angular velocity vector, with respect to inertial space, into the instantaneous direction of the OX, OY, and OZ axes to obtain P, Q, and R, respectively (see Figure 1-2, p. 15). It should be recalled that P, Q, and R are the components of the total angular velocity of the body (aircraft) with respect to inertial space. Thus, they are the angular velocities that would be measured by rate gyros fixed to these axes. It should also be recalled that inertial space is that space where Newton's laws apply. In general, a set of axes with their origin at the center of the earth but not rotating with the earth may be considered as an inertial coordinate system. Thus, the earth rotates once a day with respect to such an axis system.

1-3 DEVELOPMENT OF THE EQUATIONS OF MOTION (CONTROLS LOCKED)

The equations of motion for the aircraft can be derived from Newton's second law of motion, which states that the summation of all external forces acting on a body must be equal to the time rate of change of its momentum, and the summation of the external moments acting on a body must be equal to the time rate of change of its moment of momentum (angular momentum). The time rates of change are all taken with respect to inertial space. These laws can be expressed by two vector equations,

$$\sum \mathbf{F} = \frac{d}{dt}(m\mathbf{V}_T)\Big]_I \tag{1-1}$$

and

$$\sum \mathbf{M} = \frac{d\mathbf{H}}{dt}\Big]_I \tag{1-2}$$

where $]_I$ indicates the time rate of change of the vector with respect to inertial space. Rigorously Eq. 1-1 can be applied only to a constant-mass system. For systems with large mass variations, such as rockets, Lagrange's

equation (see Chapter 11) must be used; thus the constant-mass assumption discussed in the next paragraph. Now, the external forces and moments consist of equilibrium or steady-state forces and moments and changes in them which cause or result in a disturbance from this steady state or equilibrium condition. Thus,

$$\sum \mathbf{F} = \sum \mathbf{F}_0 + \sum \Delta\mathbf{F}$$

and

$$\sum \mathbf{M} = \sum \mathbf{M}_0 + \sum \Delta\mathbf{M} \tag{1-3}$$

where $\Sigma\mathbf{F}_0$ and $\Sigma\mathbf{M}_0$ are the summations of the equilibrium forces and moments. In the dynamic analyses to follow, the aircraft is always considered to be in equilibrium before a disturbance is introduced. Thus, $\Sigma\mathbf{F}_0$ and $\Sigma\mathbf{M}_0$ are identically zero. The equilibrium forces consist of lift, drag, thrust, and gravity, and the equilibrium moments consist of moments resulting from the lift and drag generated by the various portions of the aircraft and the thrust. Therefore, the aircraft is initially in unaccelerated flight, and the disturbances in general arise from either control surface deflections or atmospheric turbulence. Under these conditions, Eqs. 1-1 and 1-2 can be written in the form of

$$\sum \Delta\mathbf{F} = \frac{d}{dt}(m\mathbf{V}_T)\Big]_I \tag{1-4}$$

and

$$\sum \Delta\mathbf{M} = \frac{d\mathbf{H}}{dt}\Big]_I \tag{1-5}$$

Before proceeding with the derivation, it is necessary to make some additional assumptions:

2. *The mass of the aircraft remains constant during any particular dynamic analysis.* Actually, there is considerable difference in the mass of an aircraft with and without fuel, but the amount of fuel consumed during the period of the dynamic analysis may be safely neglected.
3. *The aircraft is a rigid body.* Thus, any two points on or within the airframe remain fixed with respect to each other. This assumption greatly simplifies the equations and is quite valid for fighter type aircraft. The effects of aeroelastic deflection of the airframe will be discussed in Chapter 11.
4. *The earth is an inertial reference, and unless otherwise stated the atmosphere is fixed with respect to the earth.* Although this assumption is

invalid for the analysis of inertial guidance systems, it is valid for analyzing automatic control systems for both aircraft and missiles, and it greatly simplifies the final equations. The validity of this assumption is based upon the fact that normally the gyros and accelerometers used for control systems are incapable of sensing the angular velocity of the earth or accelerations resulting from this angular velocity such as the Coriolis acceleration.

It is now time to consider the motion of an aircraft with respect to the earth. Equation 1-4 can be expanded to obtain

$$\sum \Delta \mathbf{F} = m \left. \frac{d\mathbf{V}_T}{dt} \right]_I \tag{1-6}$$

As the mass is considered constant, and using the fourth assumption, Eq. 1-6 reduces to

$$\sum \Delta \mathbf{F} = m \left. \frac{d\mathbf{V}_T}{dt} \right]_E \tag{1-7}$$

It is necessary to obtain an expression for the time rate of change of the velocity vector with respect to the earth. This process is complicated by the fact that the velocity vector may be rotating while it is changing in magnitude. This fact leads to the expression for the total derivative of a vector given below (see Appendix A)

$$\left. \frac{d\mathbf{V}_T}{dt} \right]_E = \mathbf{1}_{V_T} \frac{dV_T}{dt} + \boldsymbol{\omega} \times \mathbf{V}_T \tag{1-8}$$

where $\mathbf{1}_{V_T}(dV_T/dt)$ is the change in the linear velocity, $\boldsymbol{\omega}$ is the total angular velocity of the aircraft with respect to the earth, and $\times$ signifies the cross product. $\mathbf{V}_T$ and $\boldsymbol{\omega}$ can be written in terms of their components, so that

$$\mathbf{V}_T = \mathbf{i}U + \mathbf{j}V + \mathbf{k}W \tag{1-9}$$

and

$$\boldsymbol{\omega} = \mathbf{i}P + \mathbf{j}Q + \mathbf{k}R \tag{1-10}$$

where **i**, **j**, and **k** are unit vectors along the aircraft's X, Y, and Z axes, respectively. Then from Eq. 1-8

$$\mathbf{1}_{V_T} \frac{dV_T}{dt} = \mathbf{i}\dot{U} + \mathbf{j}\dot{V} + \mathbf{k}\dot{W} \tag{1-11}$$

and

$$\boldsymbol{\omega} \times \mathbf{V}_T = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ U & V & W \end{vmatrix} \tag{1-12}$$

Expanding,

$$\boldsymbol{\omega} \times \mathbf{V}_T = \mathbf{i}(WQ - VR) + \mathbf{j}(UR - WP) + \mathbf{k}(VP - UQ) \tag{1-13}$$

$\Sigma \Delta \mathbf{F}$ can be written in terms of its components as follows:

$$\sum \Delta \mathbf{F} = \mathbf{i} \sum \Delta F_x + \mathbf{j} \sum \Delta F_y + \mathbf{k} \sum \Delta F_z \tag{1-14}$$

Equating the components of Eqs. 1-14, 1-11, and 1-13, the equations of linear motion are obtained:

$$\begin{aligned} \sum \Delta F_x &= m(\dot{U} + WQ - VR) \\ \sum \Delta F_y &= m(\dot{V} + UR - WP) \\ \sum \Delta F_z &= m(\dot{W} + VP - UQ) \end{aligned} \tag{1-15}$$

To obtain the equations of angular motion, it is necessary to return to Eq. 1-5, which is repeated here:

$$\sum \Delta \mathbf{M} = \left. \frac{d\mathbf{H}}{dt} \right]_I \tag{1-16}$$

Before proceeding, it is necessary to obtain an expression for **H**. By definition, **H** is the angular momentum, or moment of momentum, of a revolving body. The momentum of the element of mass dm due to the angular velocity $\boldsymbol{\omega}$ will be equal to the tangential velocity of the element of mass about the instantaneous center of rotation times dm. The tangential velocity can be expressed by the vector cross product as follows (see Figure 1-1):

$$\mathbf{V}_{\tan} = \boldsymbol{\omega} \times \mathbf{R} \tag{1-17}$$

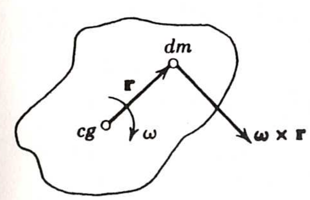

Figure 1-1 General body with an angular velocity ω about its center of gravity.

Then the incremental momentum resulting from this tangential velocity of the element of mass can be expressed as

$$d\mathbf{M} = (\boldsymbol{\omega} \times \mathbf{r})\, dm \tag{1-18}$$

The moment of momentum is the momentum times the lever arm, or, as a vector equation,

$$d\mathbf{H} = \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm \tag{1-19}$$

But $\mathbf{H} = \int d\mathbf{H}$ over the entire mass of the aircraft. Thus

$$\mathbf{H} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm \tag{1-20}$$

In evaluating the triple cross product, if

$$\boldsymbol{\omega} = \mathbf{i}P + \mathbf{j}Q + \mathbf{k}R$$

and

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

then

$$\boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ x & y & z \end{vmatrix} \tag{1-21}$$

Expanding,

$$\boldsymbol{\omega} \times \mathbf{r} = \mathbf{i}(zQ - yR) + \mathbf{j}(xR - zP) + \mathbf{k}(yP - xQ) \tag{1-22}$$

Then

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ zQ - yR & xR - zP & yP - xQ \end{vmatrix} \tag{1-23}$$

Expanding,

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \mathbf{i}\left[(y^2 + z^2)P - xyQ - xzR\right] + \mathbf{j}\left[(z^2 + x^2)Q - yzR - xyP\right] + \mathbf{k}\left[(x^2 + y^2)R - xzP - yzQ\right] \tag{1-24}$$

Substituting Eq. 1-24 into Eq. 1-20, it becomes

$$\begin{aligned} \mathbf{H} = & \int \mathbf{i}\left[(y^2 + z^2)P - xyQ - xzR\right] dm \\ & + \int \mathbf{j}\left[(z^2 + x^2)Q - yzR - xyP\right] dm \\ & + \int \mathbf{k}\left[(x^2 + y^2)R - xzP - yzQ\right] dm \end{aligned} \tag{1-25}$$

But $\int (y^2 + z^2)\, dm$ is defined to be the moment of inertia I_x, and $\int xy\, dm$ is defined to be the product of inertia J_{xy}. The remaining integrals of Eq. 1-25 are similarly defined. By remembering from the first assumption that $J_{xy} = J_{yz} = 0$, Eq. 1-25 can be rewritten in component form as

$$\begin{aligned} H_x &= PI_x - RJ_{xz} \\ H_y &= QI_y \\ H_z &= RI_z - PJ_{xz} \end{aligned} \tag{1-26}$$

However, Eq. 1-16 indicates that the time rate of change of **H** is required. As **H** can change in magnitude and direction, Eq. 1-16 can be written as

$$\sum \Delta\mathbf{M} = \mathbf{1}_H \frac{dH}{dt} + \boldsymbol{\omega} \times \mathbf{H} \tag{1-27}$$

The components of $\mathbf{1}_H\, dH/dt$ are

$$\begin{aligned} \frac{dH_x}{dt} &= \dot{P}I_x - \dot{R}J_{xz} \\ \frac{dH_y}{dt} &= \dot{Q}I_y \\ \frac{dH_z}{dt} &= \dot{R}I_z - \dot{P}J_{xz} \end{aligned} \tag{1-28}$$

As the aircraft is assumed to be a rigid body of constant mass, the time rates of change of the moments and products of inertia are zero. Now,

$$\boldsymbol{\omega} \times \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ H_x & H_y & H_z \end{vmatrix} \tag{1-29}$$

Expanding,

$$\boldsymbol{\omega} \times \mathbf{H} = \mathbf{i}(QH_z - RH_y) + \mathbf{j}(RH_x - PH_z) + \mathbf{k}(PH_y - QH_x) \qquad (1\text{-}30)$$

Also $\sum \Delta \mathbf{M}$ can be written as

$$\sum \Delta \mathbf{M} = \mathbf{i} \sum \Delta \mathscr{L} + \mathbf{j} \sum \Delta \mathscr{M} + \mathbf{k} \sum \Delta \mathscr{N} \qquad (1\text{-}31)$$

By equating components of Eqs. 1-28, 1-30, and 1-31 and substituting for H_x, H_y, and H_z from Eq. 1-26, the angular equations of motion are obtained:

$$\sum \Delta \mathscr{L} = \dot{P} I_x - \dot{R} J_{xz} + QR(I_z - I_y) - PQJ_{xz}$$

$$\sum \Delta \mathscr{M} = \dot{Q} I_y + PR(I_x - I_z) + (P^2 - R^2) J_{xz}$$

$$\sum \Delta \mathscr{N} = \dot{R} I_z - \dot{P} J_{xz} + PQ(I_y - I_x) + QRJ_{xz} \qquad (1\text{-}32)$$

The equations of linear motion from Eq. 1-15 are

$$\sum \Delta F_x = m(\dot{U} + WQ - VR)$$

$$\sum \Delta F_y = m(\dot{V} + UR - WP)$$

$$\sum \Delta F_z = m(\dot{W} + VP - UQ) \qquad (1\text{-}33)$$

Equations 1-32 and 1-33 are the complete equations of motion for the aircraft. It will next be necessary to linearize the equations and expand the left-hand sides.

Summary of Nomenclature

Axis	Direction	Name	Linear Velocity	Small Angular Displacement	Angular Velocity
OX	Forward	Roll	*U*	ϕ	*P*
OY	Right wing	Pitch	*V*	θ	*Q*
OZ	Downward	Yaw	*W*	ψ	*R*

Axis	Moment of Inertia	Product of Inertia	Force	Moment
OX	I_x	$J_{xy} = 0$	F_x	$\mathscr{L}$
OY	I_y	$J_{yz} = 0$	F_y	$\mathscr{M}$
OZ	I_z	$J_{zx} \neq 0$	F_z	$\mathscr{N}$

These conclusions are based on the assumptions that:

1. *OX* and *OZ* are in the plane of symmetry.
2. The mass of the aircraft is constant.
3. The aircraft is a rigid body.
4. The earth is an inertial reference.

1-4 AIRCRAFT ATTITUDE WITH RESPECT TO THE EARTH

In order to describe the motion of the aircraft with respect to the earth or inertial space, it is necessary to be able to specify the orientation of one axis system with respect to another. This can be done through the use of a set of angles called "Euler angles." Consider an earth axis system with its origin at the center of gravity of the aircraft and nonrotating with respect to the earth. Let OX_E and OY_E be in the horizontal plane, and OZ_E vertical and down. OX_E may be taken as north or any other fixed direction. Referring to Figure 1-2, let the following angles indicate the rotation of the *XYZ* axis from the

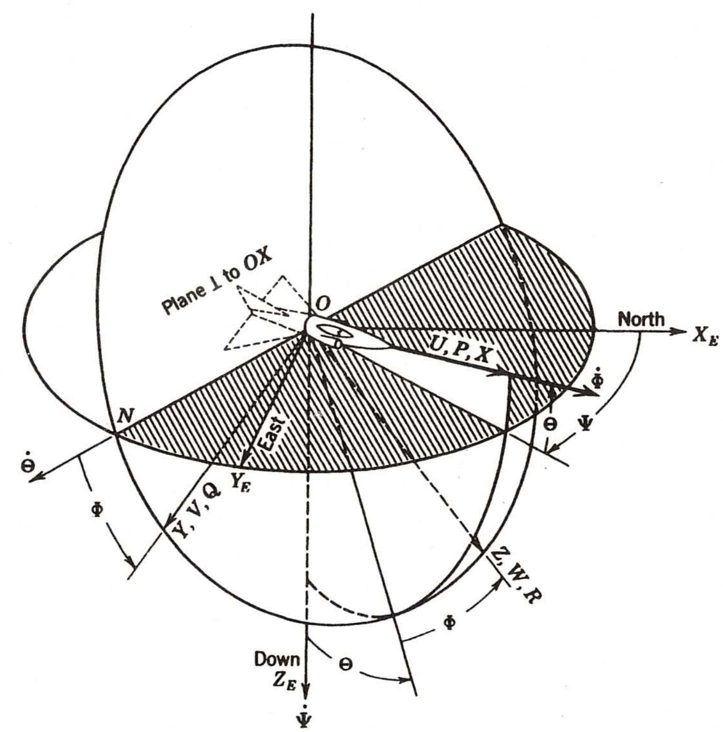

Figure 1-2 Sketch of fixed and aircraft axes.

earth axis:

Ψ is the angle between OX_E and the projection of the OX axis on the horizontal plane.

$\dot{\Psi}$ is a vector along OZ_E.

Θ is the angle between the horizontal and the OX axis measured in the vertical plane.

$\dot{\Theta}$ is a vector along ON, the line of nodes.

Φ is the angle between ON and the OY axis measured in the OYZ plane. Note that this plane is not necessarily vertical.

$\dot{\Phi}$ is a vector along OX.

Thus, the angles Ψ, Θ, and Φ specify the orientation of the aircraft axis system with respect to the earth. The positive direction of these angles is indicated in Figure 1-2.

To transform the components of the angular velocity of the aircraft from the earth axis to the aircraft axis system, take the components $\dot{\Psi}$, $\dot{\Theta}$, and $\dot{\Phi}$ and project them along the OX, OY, and OZ axes to obtain

$$
\begin{aligned}
P &= \dot{\Phi} - \dot{\Psi}\sin\Theta \\
Q &= \dot{\Theta}\cos\Phi + \dot{\Psi}\cos\Theta\sin\Phi \\
R &= -\dot{\Theta}\sin\Phi + \dot{\Psi}\cos\Theta\cos\Phi
\end{aligned}
\tag{1-34}
$$

These equations can be solved for $\dot{\Phi}$, $\dot{\Theta}$, and $\dot{\Psi}$ to yield

$$
\begin{aligned}
\dot{\Theta} &= Q\cos\Phi - R\sin\Phi \\
\dot{\Phi} &= P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta = P + \dot{\Psi}\sin\Theta \\
\dot{\Psi} &= Q\frac{\sin\Phi}{\cos\Theta} + R\frac{\cos\Phi}{\cos\Theta}
\end{aligned}
\tag{1-34a}
$$

A similar transformation can be made for linear velocities. It should be noted that $\dot{\Phi}$, $\dot{\Theta}$, and $\dot{\Psi}$ are not orthogonal vectors. Equations 1-34a can be integrated with respect to time, and by knowing the initial conditions, Θ, Φ, and Ψ can be determined; however, as the rates of change of these angles are a function of the angles themselves, this is best done on a computer.

The components of the gravity force along the aircraft axes are along

$$
\begin{aligned}
OX&: \quad -mg\sin\Theta \\
OY&: \quad mg\cos\Theta\sin\Phi \\
OZ&: \quad mg\cos\Theta\cos\Phi
\end{aligned}
\tag{1-35}
$$

1-5 LINEARIZATION AND SEPARATION OF THE EQUATIONS OF MOTION

A study of Eqs. 1-32 and 1-33 shows that it takes six simultaneous nonlinear equations of motion to completely describe the behavior of a rigid aircraft. In this form, a solution can be obtained only by the use of analog or digital computers or by manual numerical integration. In most cases, however, by the use of proper assumptions the equations can be broken down into two sets of three equations each and these linearized to obtain equations amenable to analytic solutions of sufficient accuracy.

The six equations are first broken up into two sets of three simultaneous equations. To accomplish this the aircraft is considered to be in straight and level unaccelerated flight and then to be disturbed by deflection of the elevator. This deflection applies a pitching moment about the OY axis, causing a rotation about this axis which eventually causes a change in F_x and F_z, but does not cause a rolling or yawing moment or any change in F_y; thus $P = R = V = 0$ and the $\sum \Delta F_y$, $\sum \Delta \mathscr{L}$, and $\sum \Delta \mathscr{N}$ equations may be eliminated. This leaves

$$
\left.
\begin{aligned}
\sum \Delta F_x &= m(\dot{U} + WQ) \\
\sum \Delta F_z &= m(\dot{W} - UQ) \\
\sum \Delta \mathscr{M} &= \dot{Q} I_y
\end{aligned}
\right\} \quad \text{longitudinal equations for } P = R = V = 0
\tag{1-36}
$$

An investigation of the remaining three equations, especially the $\mathscr{L}$ and $\mathscr{N}$ equations, shows that a rolling or yawing moment excites angular velocities about all three axes; thus except for certain cases the equations cannot be decoupled. The assumptions necessary for this decoupling will be discussed in Chapter 3 on the lateral dynamics of the aircraft, and the condition when this separation of the equations is not valid will be discussed in Chapter 5 on inertial cross-coupling. The rest of this chapter will be devoted to the expansion of the longitudinal equations of motion.

1-6 LONGITUDINAL EQUATIONS OF MOTION

Previously, the components of the total instantaneous values of the linear and angular velocities resolved into the aircraft axes were designated as U, V, W, P, Q, and R. As these values include an equilibrium value and the