# Propeller Speed Sensitivity to small-signal input, 'u' and subsequent nominal speed change $w_0$

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We would like to neglect the 'C' Term below in our propeller speed differential equation. Let's see how we can make a claim to do this...

## Recall our Taylor-Series linearized Equation

$$f = \frac{d\omega}{dt} = -\left(\frac{K_t\,K_b}{RJ} + \frac{2\,d\omega_0}{\eta r^3 J}\right)\omega + \frac{K_t}{RJ}u + \frac{d\omega_0^{\,2}}{\eta r^3 J}$$

From which we represent the terms as...

$$A := \left(\frac{K_t\,K_b}{RJ} + \frac{2\,d\omega_0}{\eta r^3 J}\right):$$

$$B := \frac{K_t}{R\,J}:$$

$$C := \frac{d\,\omega_0^{\,2}}{\eta\,r^3 J}:$$

## Let's Define some constants...

Figure battery voltage available will be 1-20V

**Pick a lower-bound voltage** $u_l := 5:$

**Pick an upper-bound voltage** $u_h := 15:$

Assume a motor advertized with Kv~2000 RPM/V.

**Motor Torque Constant** $K_t := 10^{-2}:$

Here is an interesting equality better explained in sources you can look up...

**Motor Back EMF Constant** $K_b := K_t:$

Assume a drag-factor order-of-magnitude from our earlier efforts.

**Drag Factor** $d := 10^{-6}:$

No gearbox, and J is common to both demoninators of the sensitivity equations below so it can be set to unity:

$$r := 1 :$$
$$\eta := 1 :$$
$$J := 1 :$$

Internal resistance of motor (safe assumption):

$$R := 1 :$$

This gets us to...

$$B = \frac{1}{100}$$
$$C = 0.06750676201$$

**We see that, 'C' is not smaller than, 'B'**...in any case magnitude of these multipliers is not what matters.

## Sensitivity Analysis

What matters is the sensitivity of $d\omega/dt$ to $B(\Delta u)$ and $C(\Delta\omega_0)$. We look at these sensitivities in turn, and determine if one dominates over the other...we hope $B(\Delta u)$ dominates over $C(\Delta\omega_0)$ because we want to neglect the, 'C' term. Let's find out!

At steady–state we have no propeller acceleration so solve for motor speed $\omega$ as function of input voltage from the differential equation for motor speed:

$$\frac{d\omega}{dt} = 0 = -\frac{K_t\, K_b}{RJ}\omega - \frac{d}{\eta r^3 J}\omega^2 + \frac{K_t}{RJ}u$$

**Rearrange and solve polynomial equation**

$$\omega := x \rightarrow \frac{-\frac{K_t \cdot K_b}{d} + \sqrt{\left(\frac{K_t \cdot K_b}{d}\right)^2 - 4 \cdot \left(-\frac{K_t}{d} \cdot x\right)}}{2} :$$

**Corresponding lower and upper steady-state propeller speeds by solving the above equation for lower- and upper-bound drive voltages...**

$$\omega_l := \ evalf\left(map\left(u \rightarrow \omega(u),\ u_l\right)\right)$$
$$\omega_l := 179.1287848 \qquad (1)$$

$$\omega_h := \ evalf\left(map\left(u \rightarrow \omega(u),\ u_h\right)\right)$$

$$\omega_h := 340.5124838 \tag{2}$$

We need an ω0 for the initial estimate for 'C' up above...here it is as the average within our range...

$$\omega_0 := \frac{(\omega_h + \omega_l)}{2}$$

$$\omega_0 := 259.8206343 \tag{3}$$

**Specify a fraction of this scale that represents, "small signal"**

$stepsize := 0.1 :$

$pfs := \langle seq(i \cdot stepsize,\ i = 1 ..10) \rangle :$

Now revisit our linearized speed equation and examine the sensitivity of propeller acceleration to "small-signal" drive input 'u' and what would be the resultant change of speed to $\omega_0$:the corresponding steady-state value for which we can estimate from above.

$$f = \frac{d\omega}{dt} = -\left(\frac{K_t \cdot K_b}{RJ} + \frac{2\, d\omega_0}{\eta r^3 J}\right)\omega + \frac{K_t}{RJ} u + \frac{d\omega_0{}^2}{\eta r^3 J}$$

$$\frac{\partial}{\partial\, \omega_0} f = -\frac{2\, d\omega}{\eta r^3 J} + \frac{2\, d\omega_0}{\eta r^3 J} = \frac{2\, d(\omega_0 - \omega)}{\eta r^3 J} = \frac{2\, d \cdot (\Delta\omega)}{\eta r^3 J}$$

Establish drive voltage steps from the specified range and step value pfs. We will consider each of these, "step sizes" to decide what to accept as acceptible, "small signal" behavior relative to sensitivity of the speed equation to drive voltage 'u' and resultant drag induced by $\omega_0$. We would like to ignore small-signal drag as we want to drop the, 'C' term.

$$\Delta u := evalf(pfs \cdot (u_h - u_l))$$

$$\Delta u := \begin{bmatrix} 1. \\ 2. \\ 3. \\ 4. \\ 5. \\ 6. \\ 7. \\ 8. \\ 9. \\ 10. \end{bmatrix} \tag{4}$$

Calculate resultant $\Delta\omega$ as the new steady–state resulting from the drive voltage step–changes above

$$\Delta\omega := evalf\left(map\left(u \rightarrow \omega(u),\ \left(u_l + \sim\Delta u\right)\right)\right) - \sim\omega_l$$

$$\Delta\omega := \begin{bmatrix} 20.8712152000000 \\ 40.1294555567252 \\ 58.0993475269015 \\ 75.0093417149110 \\ 91.0274270716424 \\ 106.281411824968 \\ 120.871215200000 \\ 134.876709664026 \\ 148.362936963537 \\ 161.383698995333 \end{bmatrix} \tag{5}$$

(the ~ symbol above is for element-by-element operations on vectors. It's a Maple software symbol. I'm using Maple software here. It's awesome.)

Sensitivity to small-signal $\omega_0$: when we multiply by small–signal $\Delta\omega$ our equation for sensitivity of the linearized motor speed equation relative to small changes in nominal propeller speed that result in drag on the differential equation becomes the following. Notice the partial derivative above gives us one $\Delta\omega$, so we square this when multiplying it by $\Delta\omega$ for the sensitivity to change in $\omega$. Hence the squared term.

$$S_\omega := \left\langle evalm\left( \frac{2\, d\, \Delta\omega^{\sim 2}}{\eta\, r^3\, J} \right) \right\rangle$$

$$S_\omega := \begin{bmatrix} 0.000871215247849422 \\ 0.00322074640655836 \\ 0.00675106836610334 \\ 0.0112528026890086 \\ 0.0165719849585664 \\ 0.0225914769990171 \\ 0.0292197013278494 \\ 0.0363834536195879 \\ 0.0440231221288932 \\ 0.0520893966028323 \end{bmatrix} \tag{6}$$

Similarly to estimate sensitivity to small-signal drive input changes take partial derivative of the linearized equation with respect to drive voltage...

$$\frac{\partial}{\partial u} f = \frac{K_t}{R\, J} :$$

Multiply by Δu for estimate of sensitivity of the $\frac{d\omega}{dt}$ : equation to command input changes.

$$S_u := evalf\left( \frac{K_t}{R\, J} \cdot \Delta u \right) =$$

$$S_u := \begin{bmatrix} 0.0100000000000000 \\ 0.0200000000000000 \\ 0.0300000000000000 \\ 0.0400000000000000 \\ 0.0500000000000000 \\ 0.0600000000000000 \\ 0.0700000000000000 \\ 0.0800000000000000 \\ 0.0900000000000000 \\ 0.100000000000000 \end{bmatrix} \tag{7}$$

The ratio of the sensitivity of acceleration to drive voltage input and resultant small-signal drag effect produced by the speed change associated with the drive change (again the ~ is a Maple software designation for element-by-element division)

$$ratio := evalf\left(\frac{S_u}{\sim S_\omega}\right)$$

$$ratio := \begin{bmatrix} 11.4782196761189 \\ 6.20974068597089 \\ 4.44374110483431 \\ 3.55466998804404 \\ 3.01714007857303 \\ 2.65586884835420 \\ 2.39564392580847 \\ 2.19880170905299 \\ 2.04438021766138 \\ 1.91977650965077 \end{bmatrix} \quad (8)$$

For a nominal 1V drive input change the ratio is near 10. For a 2-volt change it drops to 6 or-so. You can visualize this on the plot below.

This indicates that the propeller acceleration equation is 5-10 times more sensitive to voltage input changes in the 1-2 volt range than it is to the resultant drag effect.

**This drives to a conclusion that we can ignore the drag-related constant, 'C' in our simplified model required for, "classical" control design forthwith.**

## One More Adjustment...

Looking at our linearized equation, we're now comfortable neglecting the last term (our, C term). However, look at that first expression in parenthesis. the second term in there is 2–times the C term assuming $\omega \sim \omega_{0,}$ as is the case for small-signal behavior we are going to examine our controller around:

$$\omega_0 = 259.8206343$$

$\omega_1 := \omega_0 + \Delta\omega[1]$

$$\omega_1 := 280.691849500000 \quad (9)$$

For the one–volt step we'll likely use for small–signal dynamics evaluation, taking $\omega = \omega_0$ for purpose of lumping the $\omega$'s in the drag portion of the 'A' term is true within 10%:

$$\frac{\omega_0^{\ 2}}{\omega_0\,\omega_1} = 0.925643672100996$$

We now see why the 'C' term is there in the first place! After the -2 multiple in the 'A' term the 'C' simply adds-back 1 multiple of the same value. If we neglect the 'C' term we should kill that, '2' in the A term, otherwise we're doubling the effect of drag in the, "natural response" (the unforced portion of the differential equation).

$$f = \frac{d\omega}{dt} = -\left(\frac{K_t \cdot K_b}{RJ} + \frac{2\,d\omega_0}{\eta r^3 J}\right)\omega + \frac{K_t}{RJ}u + \frac{d\omega_0^{\ 2}}{\eta r^3 J}:$$

## Final linearized, simplified propeller speed equation

After neglecting the 'C' term and adjusting the 'A' term accordingly, this is the differential equation we'll use for our "classical" controller design...

$$f_{simple} = \frac{d\omega}{dt} = -\left(\frac{K_t K_b}{RJ} + \frac{d\omega_0}{\eta r^3 J}\right)\omega + \frac{K_t}{RJ}u:$$

## Humourous Observation...

Before the Taylor Series linearization our propeller speed differential equation looks like this.

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{K_t \cdot K_b}{R \cdot J}\cdot\omega - \frac{d}{\eta r^3 J}\cdot\omega^2 + \frac{K_1}{R \cdot J}\cdot u:$$

After going through all of the above we might realize we could have arrived at our simplification by simple inspection! We could have taken one $\omega$ from under the square and tossed in into that drag term as an $\omega_0$ , telling ourselves we'd "slide" it as a constant at some intervals of the $\omega$ range! This produces the $f_{simple}$: exactly.

Perhaps it is only obvious now after more mathematically rigourous treatment. In any case, revisiting the original, non-linear differential equation to compare it with where we ended-up does add to my confidence that this was fair treatment of the equation simplification.

## Closing Note...

The sensitivity graph below might temp us to think we can "step" the drive voltage half-and-more of the full range and still be able to "ignore" the effect of drag (drop the 'C' term) because we can see below it appears we'd still by 5+ times more sensitive to drive than drag.

However, our control system analysis will focus on the small-signal behavior in the 10-20% of full-scale drive, at least initially.

```
plot(pfs, ratio, labels = ["small signal fraction of full-scale", "sensitivity ratio: u/w0"],
    gridlines = true, size = [400, 400], labeldirections = [horizontal, vertical])
```