Quadrotor: Classical Roll and Pitch Axis Control

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October, 2018

Ignoring Gyroscopic effects of rigid body and propellers, and with the modified, linearized propeller speed equation we arrive at a roll-axis transfer function and design a lead compensator.

Loading Units:-Standard

$with(DynamicSystems)$:

Modelling Parameters

Motor Terms

Assume a motor advertized with Kv~2000 RPM/V.

Motor Torque Constant $K_t := 10^{-2}\,\frac{\mathrm{N\,m}}{\mathrm{A}}$:

Here is an interesting equality better explained in sources you can look up...

Motor Back EMF Constant $K_b := 10^{-2}\,\frac{\mathrm{V}}{\frac{1}{\mathrm{s}}}$:

Internal resistance of motor (safe assumption):

$$R := 1\,\frac{\mathrm{V}}{\mathrm{A}} :$$

Propeller Terms

Thrust Factor

$$b := \frac{C_t\,\rho\,\mathrm{D}_p^{\,4}}{4\,\pi^2} :$$

Drag Factor

$$d := \frac{C_p\,\rho\,\mathrm{D}_p^{\,5}}{8\,\pi^3} :$$

Where these power and torque coefficients are unitless (if you cancel terms), but resulting calculated units work out if we leave in this form

$$C_p := 0.7\,\frac{\mathrm{N\,s^2}}{\mathrm{kg\,m}} :$$

$$C_t := 0.1\,\frac{\mathrm{N\,s}^2}{\mathrm{kg\,m}}:$$

$$\rho := 1.225\,\frac{\mathrm{kg}}{\mathrm{m}^3}:$$

$$\mathrm{D}_p := 0.1\mathrm{m}:$$

This gets us to thrust factor, 'b':

$evalf(b)$

$$3.102961248\;10^{-7}\;\mathrm{N\;s}^2 \tag{1}$$

And drag factor, 'd':

$evalf(d)$

$$3.456961346\;10^{-8}\;\mathrm{N\;s}^2\;\mathrm{m} \tag{2}$$

Gearbox

No gearbox, so:

$$r := 1:$$

$$\eta := 1:$$

Body Moment of Inertia

For now model as a dumbell with 100-gram motor-propeller assemblies

$$m_m := 0.1\;\mathrm{kg}:$$

$$l_a := 0.1\mathrm{m}:$$

$$I_x := 2\;m_m l_a^{\,2}$$

$$I_x := 0.002\;\mathrm{kg\;m}^2 \tag{3}$$

Motor-Propeller Moment of Inertia

Model the Prop as a 2 gram spinning rod. This will over-estimate J_P because in fact is has most mass concentrated nearer to hub.

$$m_p := 0.002\mathrm{kg}:$$

$$J_p := \frac{1}{12}\,m_p \mathrm{D}_p^{\,2}:$$

Model spinning motor shaft and rotor as spinning cylinder. Assign half motor weight to this spinning core. That's likely too much but a safe estimate.

$m_{motor} := 0.03\,\mathrm{kg}: \quad m_{rotor} := \frac{m_{motor}}{2}: \quad r_{rotor} := 0.01\,\mathrm{m}: \qquad J_m := \frac{1}{2} m_{rotor} r_{rotor}^{\;2}:$

Let's see if motor or its propeller dominate the assembly's moment of intertia...

$$\frac{J_p}{J_m}$$

$$2.222222223 \tag{4}$$

Looks like the prop dominates, but the rotor+shaft not negligible, so lump the total as Jt...

$$J_t := J_p + J_m:$$

Initial conditions

Nominal propeller speed at hover

$$\omega_0 := 100\,\mathrm{s}^{-1}:$$

Motor Speed Equation Parameters

$A := convert\left(\left(\frac{K_t \cdot K_b}{R \cdot J_t} + \frac{d\,\omega_0}{\eta \cdot r^3 \cdot J_t}\right), unit\_free\right)$

$$A := 42.80977710 \tag{5}$$

$B := convert\left(\frac{K_t}{R\,J_t}, unit\_free\right)$

$$B := 4137.931034 \tag{6}$$

'A' term relative assessment: motor parameters or drag term?

$$\frac{\frac{K_t \cdot K_b}{R \cdot J_t}}{\frac{d\,\omega_0}{\eta \cdot r^3 \cdot J_t}}$$

$$28.92713860 \tag{7}$$

Although confusing, if you expand the units and recall that 'A' multiplies an ω and 'B' multiplies command voltage input, then each term contributes a $\frac{1}{s^2}$: as we expect for the differential equation.

Transfer Function for Roll or Pitch

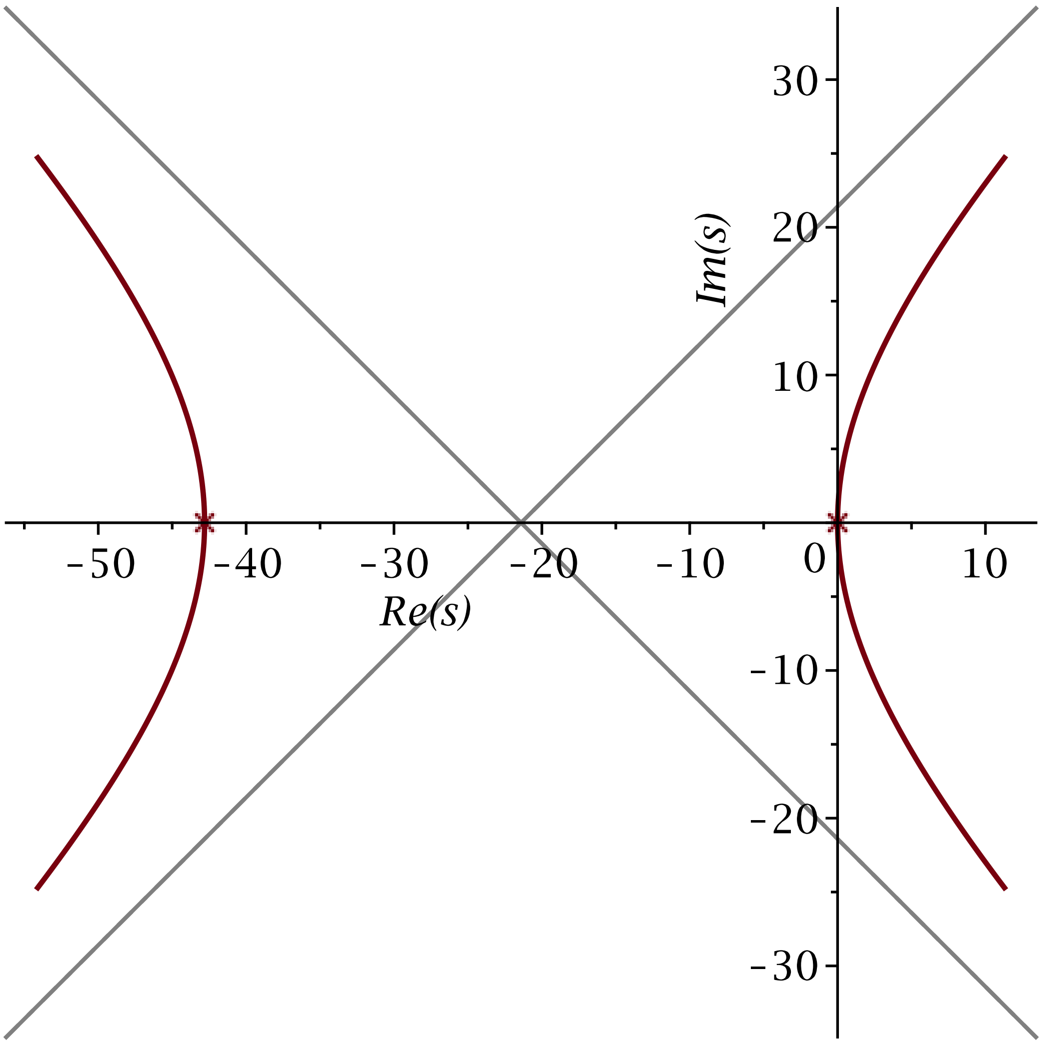
Plant gain

$$K_g := convert\left(\frac{l_a\, b\, B^2}{I_x}, unit\_free\right)$$

$$K_g := 265.6518546 \tag{8}$$

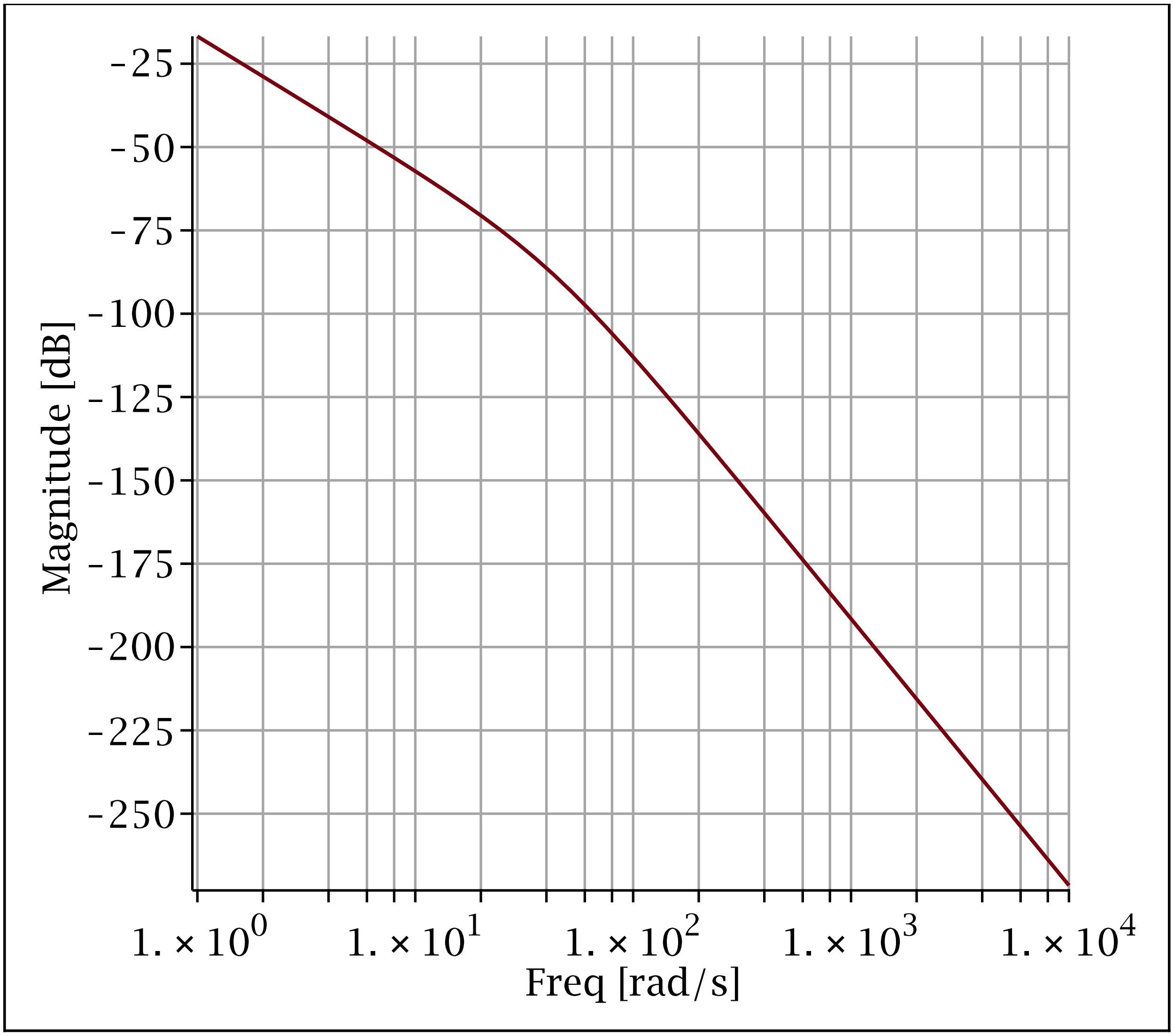
$$G := TransferFunction\left(\frac{K_g}{s^2 \cdot (s + A)^2}\right):$$

$$RootLocusPlot(G, 0\,..10000)$$

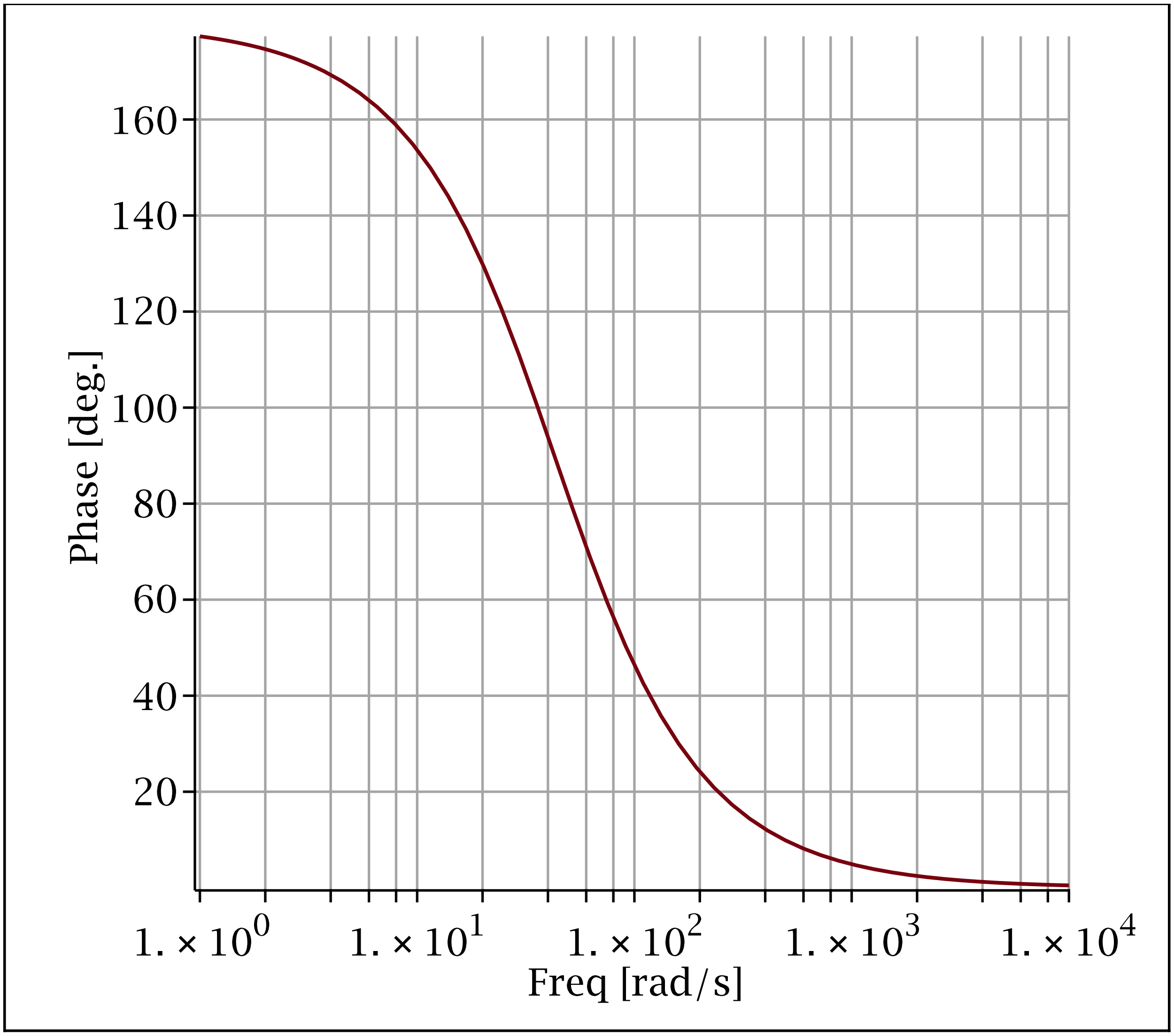


We can see above that our double-pole at zero is immediately unstable, and worse-so with any gain as the loci head-off into the righ-half plane of the proportional gain (Plant) root locus. We need to stablaize the platform.

$$BodePlot(G)$$



-25
-50
-75
-100
-125
-150
-175
-200
-225
-250
Magnitude [dB]
$1. \times 10^0$
$1. \times 10^1$
$1. \times 10^2$
$1. \times 10^3$
$1. \times 10^4$
Freq [rad/s]



Plant is unstable: Need a Lead Compensator

The plant transfer function exhibits a classic, "double integrator" for the platform subject to thrust from the opposing propellers. To control such a plant, a lead compensator is employed to provide a phase-margin, "hump" at a desired, practical unity gain crossover frequency.

"Practical" can be a bit vaque, for example a textbook problem for satellite attitude control typically presents a space capsule with thrusters. The thrusters are not typically modelled, so we solve the textbook problem assuming we have immediate thrust available to actuate the capsule. I remember this problem from school, and I also remember wondering where to pick the unity-gain crossover frequency and then my lead zero location on the frequency axis.

In our case, the, "practical" limit for our lead compensating zero is going to be below the double-pole at our, 'A' term. The 'A' term is the, "practical" limit in our case. It represents the frequency response limits of our actuating propeller pair.

Theoretically, the most pahse we can gain with a lead if 90-degrees, before we cancel it with a higher-frequency pole. However, you can see above that the double-pole at A is lowering our already zero phase-margin plant beyond any phase angle we can recover with a lead compensator.

We must place our lead zero far enough below, 'A', and a unity-gain crossover to create a phase-margin hump before the double-pole at 'A' kills the phase beyond our ability to gain any with the lead zero.

Scale the lead zero frequency down-frequency from the 'A' double-pole

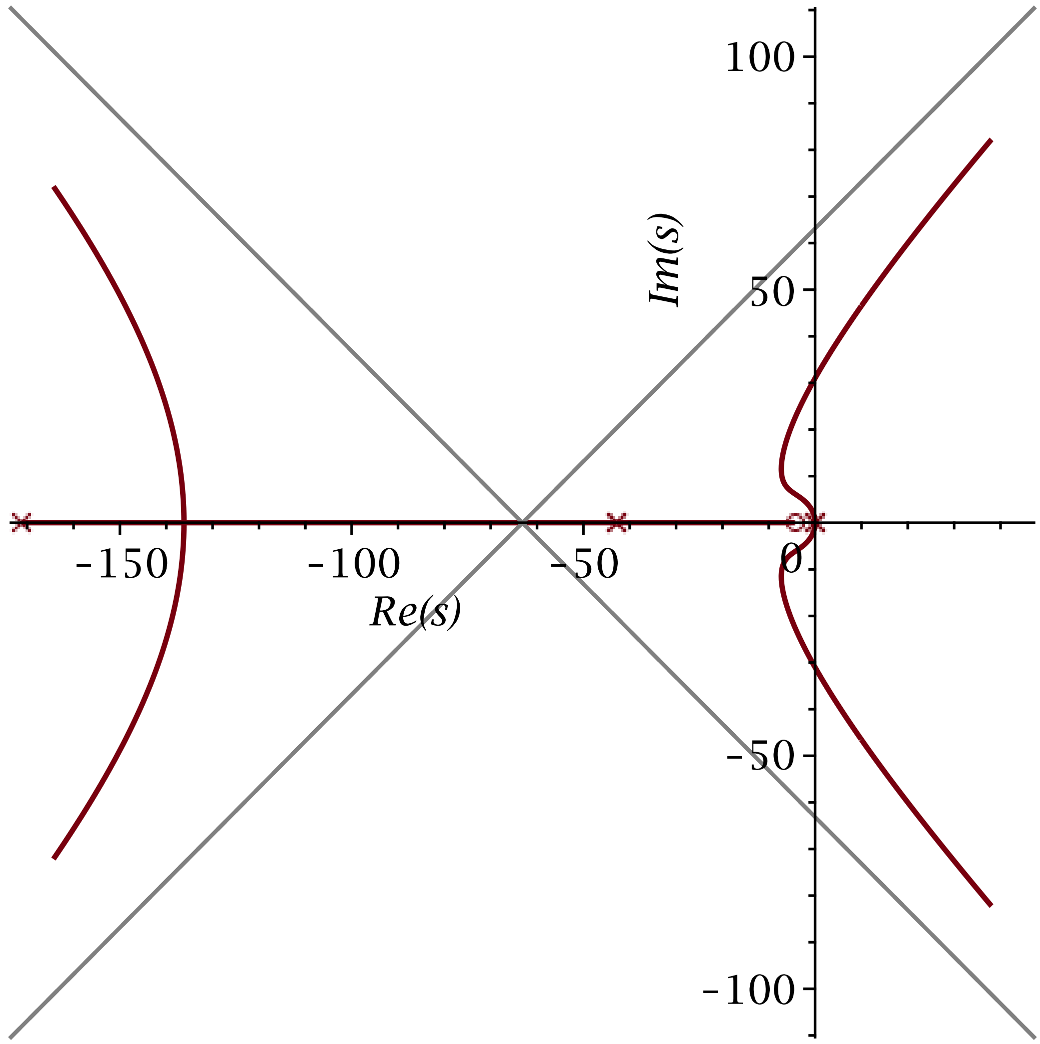
$lz := \frac{A}{10}$:

$lp := 4 \cdot A$:

Compare the resulting root-locus for the compensated plant with the plant root-locus above. You can see the effect of the zero: It's, "pulling" the loci to the left-half plane for a range of relatively low gain. Sketching rules for root-locus diagrams tell us we could pick a spot on the bulge of the root locus near the origin and estimate frequency response, phase margin and/or time-domain characteristics to expect.

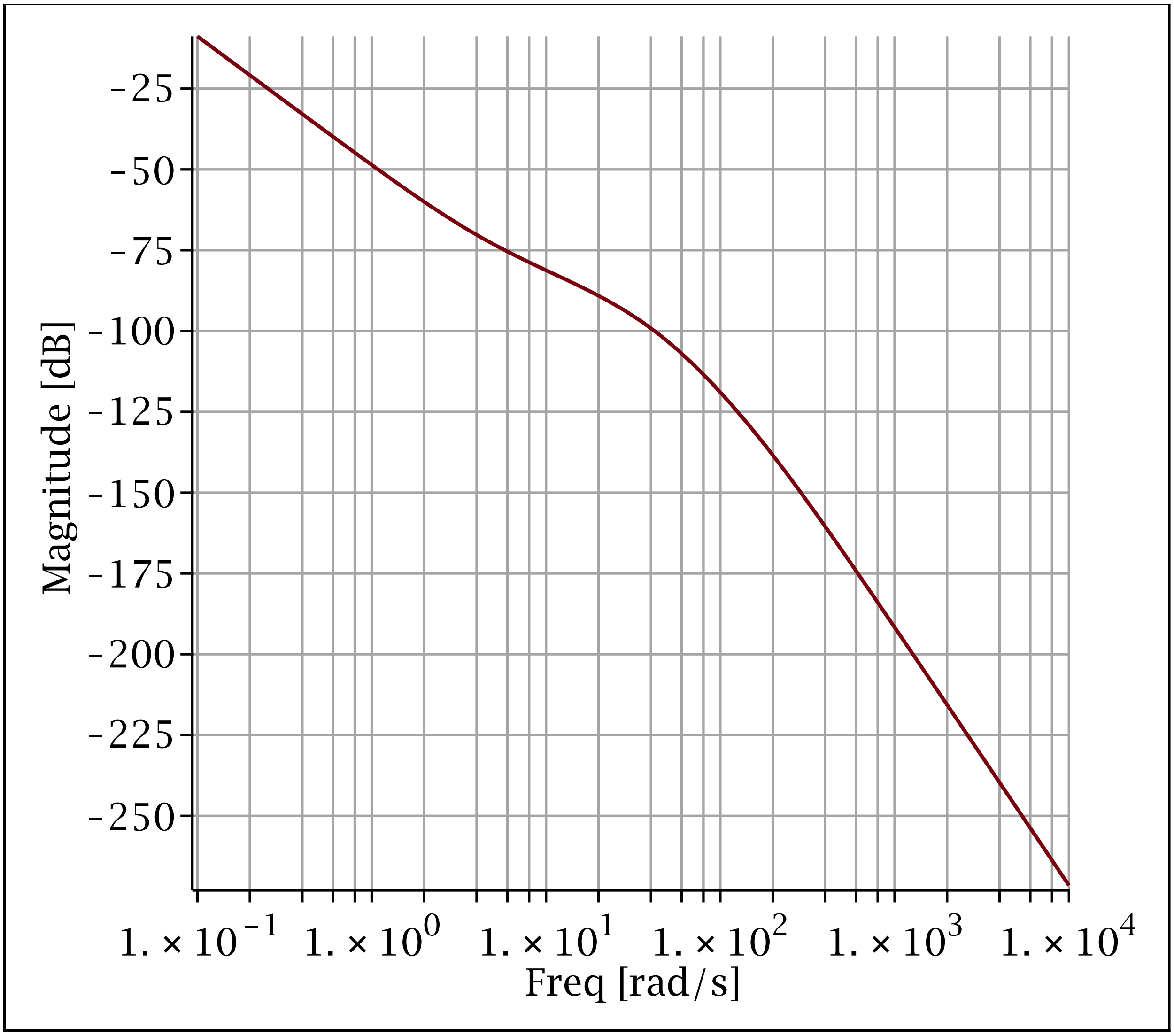
$$G_c := TransferFunction\left(\frac{s+lz}{s+lp} \cdot \frac{K_g}{s^2 \cdot (s+A)^2}\right):$$

$$RootLocusPlot(G_c, 0..1000000)$$



the Bode plot will help determine what our compensator gain needs should be. We want unity-gain crossover where our phase-margin hump peaks near 10 rad/sec.

$BodePlot(G_c)$



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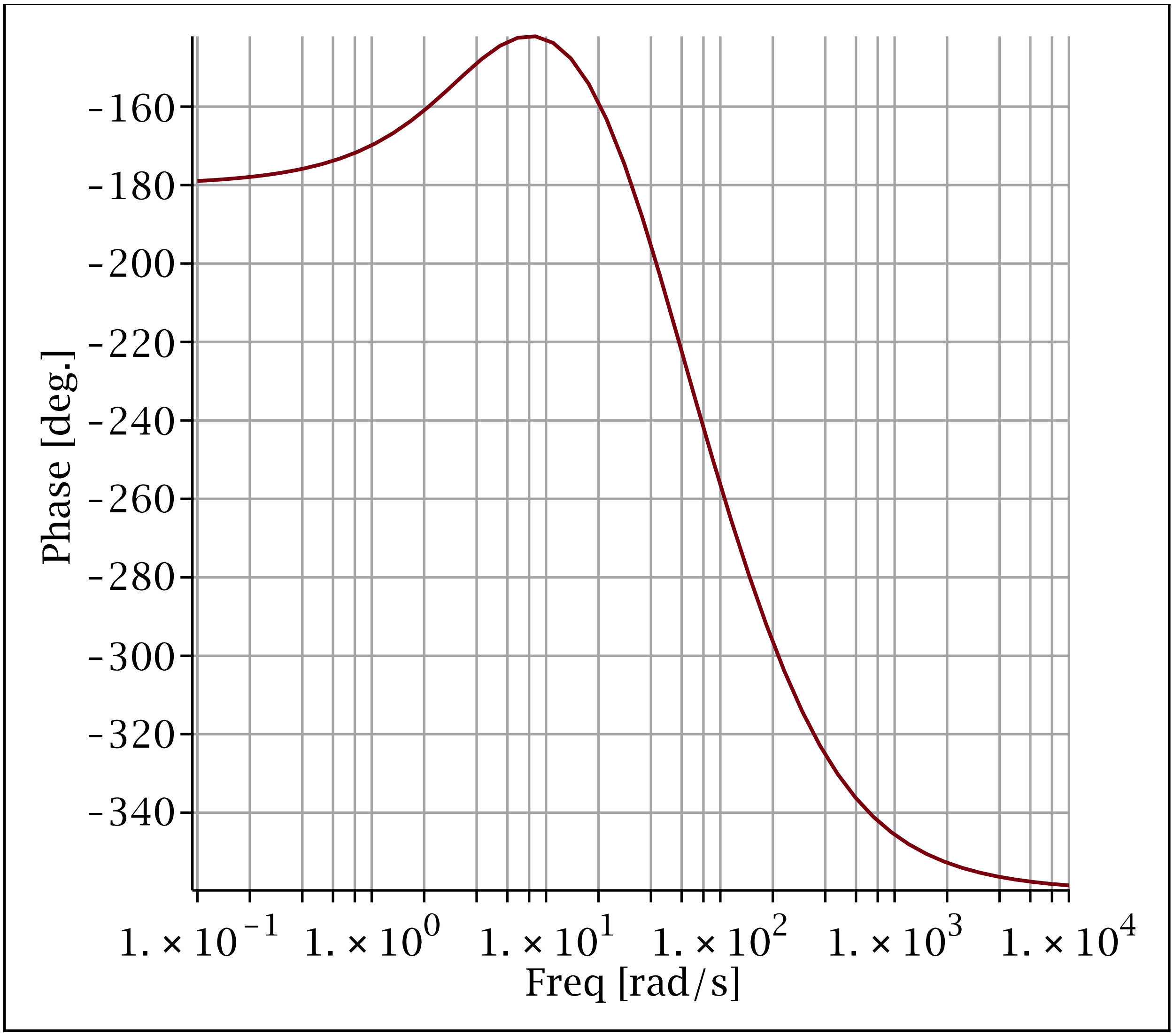
Freq [rad/s]



$BodePlot(G_c) : myplot := \% :$

$myData := plottools{:-}getdata(myplot)[-1]$